A LaTeX Theory of Equations

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Chapter 1

Basics of Polynomial equations

In algebra, the theory of equations is the study of algebraic equations, also called "polynomial equations". The main objective of the theory of equations is to find the roots of equations. In this course we solve higher degree polynomial equations, study the relations between roots and coefficients, reciprocal equations, Also we find approximate roots of equations and solve cubic and biquadratic equations.

1.1 Definitions

Definition 1.1.1 (Polynomial)**:** An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ where $a_0,\ a_1,\ a_2, \cdots, a_n$ are real numbers, $a_0 \neq 0$ and n is a non-negative integer is called a polynomial in x of degree n.

- $a_0,\ a_1,\ a_2), \cdots, a_n$ are called the coefficients of x^n , x^{n-1} , $\cdots$ and x respectively. The term a_n is called the constant term.
- $f(x) = a_0x^n + a_1x^{n-1} + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ is called a polynomial whereas $a_0x^n + a_1x^{n-1} + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ is called an equation of degree n if $a_0 \neq 0$.
- An equation of degree 1 is called a linear equation.$[ax + b = 0]$
- An equation of degree 2 is called a quadratic equation.$[ax^2 + bx + c = 0]$
- An equation of degree 3 is called a cubic equation.$[ax^3 + bx^2 + cx + d = 0]$
- An equation of degree 4 is called a biquadratic equation.$[ax^4 + bx^3 + cx^2 + dx + e = 0]$

Definition 1.1.2 (Root of an equation)**:** A number, real or imaginary a is called a root of the equation $f(x) = 0$ if $f(a)$ =0. In that case, we say $x - a$ is a factor of the polynomial $f(x)$. We can also say, a is a zero of the polynomial $f(x)$.

We call an equation all of whose coefficients are real as a *real equation*. Through out this course, by equation we mean *real equation* only.

Some important theorems

Theorem 1.1.1 (Division Algorithm)**:** When a polynomial $f(x)$ is divided by another polynomial $g(x)$, there exists polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)g(x) + r(x)$ where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$.

$f(x)$, the polynomial that is being divided is called the dividend, $g(x)$, that is being divided by is called the divisor, $q(x)$ is called the quotient and $r(x$ is called the reminder.

Theorem 1.1.2 (Remainder theorem)**:** If a polynomial $f(x)$ be divided by $x - c$ until a remainder independent of x is obtained, this remainder is equal to $f(c)$.

If r is the reminder and the $q(x)$ is the quotient when $f(x)$ is divided by $x - c$, then $f(x) = (x - c)q(x) + r$. When $x = c$, we obtain $f(c) = r$. If $r = 0$, the division is exact. Hence we have proved also the following useful theorem.

Theorem 1.1.3 (The Factor Theorem)**:** If $f(c)$ is zero, the polynomial $f(x)$ has the factor $x - c$. In other words, if c is a root of $f(x) = 0$, $x - c$ is a factor of $f(x)$.

Problem 1.1.1: Show that 3 is a root of the equation $x^3 - 4x^2 + 2x + 3 = 0$.

Solution: Let $p(x) = x^3 - 4x^2 + 2x + 3$. Put $x = 3$. We get

$$\begin{aligned} p(3) &= 3^3 - 4 \times 3^2 + 2 \times 3 + 3 \\ &= 27 - 36 + 6 + 3 = 0 \end{aligned}$$

Therefore, 3 is a root of the equation $x^3 - 4x^2 + 2x + 3 = 0$ and $x - 3$ is a factor of the polynomial $p(x) = x^3 - 4x^2 + 2x + 3$.

Definition 1.1.3 (Irrational roots:)**:** If a real number β is not perfet square then $\sqrt{\beta}$ is irrational. Generally a real number can be written as $\alpha + \sqrt{\beta}$. The number $\alpha - \sqrt{\beta}$ is called the conjugate of $\alpha + \sqrt{\beta}$.

Remark 1.1.1: Note that if both the terms of the given root are irrational then the degree of the equation is 4.

Problem 1.1.2: Show that $\sqrt{2}$ is a root of the equation $x^2 - 2 = 0$.

Solution: Let $x^2 - 2 = 0$. Put $x = \sqrt{2}$. We get

$$p(\sqrt{2}) = (\sqrt{2})^2 - 2 = 2 - 2 = 0$$

Hence $\sqrt{2}$ is a root of $x^2 - 2 = 0$. **Note:** Observe that $-\sqrt{2}$ is also a root of $p(x) = x^2 - 2 = 0$. Also note that the roots are real, but they are irrational.

Imaginary roots:

Consider the equation $x^2 + 4 = 0$. We know that x^2 is positive for all values of real numbers and hence $x^2 + 4$ never be zero for any real value of x. Hence the equation $x^2 + 4 = 0$ does not have real root. But, it has imaginary roots. The simple equation which does not have a real root is $x^2 + 1 = 0$, because there is no real number whose square is -1. Equivalently, $\sqrt{-1}$ is not real. Since $\sqrt{-1}$ is imaginary we write it as $\sqrt{-1} = i$. Note that $i^2 = -1$. Note: $\sqrt{-9} = \sqrt{9 \times -1} = \sqrt{9} \cdot \sqrt{-1} = 3i$ Complex number (or) Imaginary number: A number of the form $a + ib$ where a and b are real numbers is called a complex number. a is called the real part and b is called the imaginary part.

- If a complex number has no real part then it is called purely imaginary number.
- $a - ib$ is called the conjugate of $a + ib$.

Problem 1.1.3: Show that $2 + i$ is a root of the equation $x^2 - 4x + 5 = 0$.

Solution: Let $f(x) = x^2 - 4x + 5 = 0$.

$$f(2 + i) = (2 + i)^2 - 4(2 + i) + 5$$

$$= 2^2 + 4i + i^2 - 8 - 4i + 5$$
$$= 4 + 4i - 1 - 8 - 4i + 5$$
$$= 0$$

Hence $2 + i$ is a root of $x^2 - 4x + 5 = 0$.

Note: Observe that $2 - i$ is also a root of $x^2 - 4x + 5 = 0$.

Definition 1.1.4 (Multiple roots)**:** α is a root of $f(x) = 0$ of multiplicity m if $f(x)$ is exactly divisible by $(x - \alpha)^m$, but not by $(x - \alpha)^{m+1}$.

In other words, Let $f(x) = 0$ be an equation of degree n. α is a root of $f(x) = 0$ of multiplicity m if $f(x) = (x - \alpha)^m g(x)$ where $g(x)$ is a polynomial of degree $n - r$ and α is not a root of $g(x) = 0$ [that is, $g(\alpha) \neq 0$].

Theorem 1.1.4: Every polynomial equation $f(x) = 0$ of degree $n > 0$ has at least one root, real or imaginary(complex).

The next theorem is a consequence of the Fundamental theorem of algebra.

Theorem 1.1.5: Every polynomial equation $f(x) = 0$ of degree $n > 0$ has exactly n roots, a root of multiplicity r is counted as r roots.

Formation of equations with given roots:

The least degree equation with roots $\alpha_1, \alpha_2, \cdots \alpha_n$ is $(x - \alpha_1)(x - \alpha_1)\cdots(x - \alpha_n) = 0$.

Chapter 2

Equations with Irrational and Imaginary roots

Theorem 2.0.6: For any equation with rational coefficients, irrational roots occur in pair. In other words, Let $f(x) = 0$ be an equation with real coefficients. If $\alpha + \sqrt{\beta}$ where β is not a perfect square is a root of $f(x) = 0$, then $\alpha - \sqrt{\beta}$ is also a root.

Theorem 2.0.7: For any equation with real coefficients, imaginary roots occur in pair. In other words, Let $f(x) = 0$ be an equation with real coefficients. If $\alpha + i\beta$ is a root of $f(x) = 0$, then $\alpha - i\beta$ is also a root.

2.1 Formation of Equations with given Irrational / Imaginary roots

Problem 2.1.1: Form the equation of the lowest degree with rational coefficients one of whose roots is $2 + \sqrt{5}$.

Solution: Let $x = 2 + \sqrt{5}$. This is an equation with irrational coefficient. So, we should remove the irrational number as explained below.

$$\begin{aligned} x = 2 + \sqrt{5} \Rightarrow x - 2 &= \sqrt{5} && \text{[Keep the irrational number in one side]} \\ \Rightarrow (x-2)^2 &= (\sqrt{5})^2 && \text{[Square both sides]} \\ \Rightarrow x^2 - 4x + 4 &= 5 && \text{[Simplify]} \\ \Rightarrow x^2 - 4x - 1 &= 0 \end{aligned}$$

The required polynomial is $x^2 - 4x - 1 = 0$.

This problem can also be solved by using the theorm 2.0.6.

Since $2 + \sqrt{5}$ is a root its conjugate $2 - \sqrt{5}$ is also a root of the required equation. Hence the required lowest degree equation is

$$[x - (2 + \sqrt{5})][x - (2 - \sqrt{5})] = 0$$
$$\Rightarrow [(x-2) + \sqrt{5}][(x-2) - \sqrt{5}] = 0$$
$$\Rightarrow (x-2)^2 - (\sqrt{5})^2 = 0$$
$$\Rightarrow x^2 - 4x + 4 - 5 = 0 \text{[Using } (a+b)(a-b) = a^2 - b^2\text{]}$$
$$\Rightarrow x^2 - 4x - 1 = 0$$

We get the same answer. But the second method is time consuming when the given root is of the form $\sqrt{\alpha} + \sqrt{\alpha}$. So we follow the first method for finding the lowest degree equation.

Remark 2.1.1: Note that if one term is rational and another is irrational then the degree of the equation is 2.

Problem 2.1.2: Form the equation of the lowest degree with rational coefficients one of whose roots is $3\sqrt{5}-2\sqrt{3}$.

Solution: Let $x=2+\sqrt{5}$. This is an equation with irrational coefficient.
So, we should remove the irrational numbers as explained below.

$$
\begin{aligned}
x=3\sqrt{5}-2\sqrt{3} &\Rightarrow x^2=(3\sqrt{5}-2\sqrt{3})^2 && \text{[Square both sides]}\\
&\Rightarrow x^2=(3\sqrt{5})^2-12\sqrt{15}+(2\sqrt{3})^2 && \text{[Simplify]}\\
&\Rightarrow x^2=45-12\sqrt{15}+12 && \text{[Simplify]}\\
&\Rightarrow x^2-57=-12\sqrt{15} &&\\
&\Rightarrow (x^2-57)^2=(-12\sqrt{15})^2 && \text{[Square both sides]}\\
&\Rightarrow x^4-114x^2+3249=144\times 15 && \text{[Simplify]}\\
&\Rightarrow x^4-114x^2+3249=2160 && \text{[Simplify]}\\
&\Rightarrow x^4-114x^2+1089=0 && \text{[Simplify]}
\end{aligned}
$$

The required polynomial is $x^4-114x^2+1089=0$.

Remark 2.1.2: Note that if both the terms of the given root are irrational then the degree of the equation is 4.

Problem 2.1.3: Form the lowest degree equation with rational coefficients one of whose roots is $3-4i$.

Solution: Let $x=3-4i$. This is an equation with imaginary coefficient.
So, we should remove the imaginary number as explained below.

$$
\begin{aligned}
x=3-4i &\Rightarrow x-3=-4i &&\\
&\Rightarrow (x-3)^2=(-4i)^2 && \text{[Square both sides]}\\
&\Rightarrow x^2-6x+9=16i^2 && \text{[Simplify]}\\
&\Rightarrow x^2-6x+9=16(-1) && [i^2=-1]\\
&\Rightarrow x^2-6x+9+16=0 &&\\
&\Rightarrow x^2-6x+25=0 &&
\end{aligned}
$$

The required equation is $x^2-6x+25=0$.

Problem 2.1.4: Form the lowest degree equation with rational coefficients whose roots are $4\sqrt{3}$ and $5+2i$.

Solution: Since $4\sqrt{3}$ is a root, its conjugate $-4\sqrt{3}$ is also a root. Since $5+2i$ is a root, its complex conjugate $5-2i$ is also a root. Hence, the required equation is

$$
\begin{aligned}
&\left(x-4\sqrt{3}\right)\left(x+4\sqrt{3}\right)(x-(5+2i))(x-(5-2i))=0 &&\\
&\left(x^2-(4\sqrt{3})^2\right)((x-5)-2i)((x-5)+2i)=0 && [(a+b)(a-b)=a^2-b^2]\\
&\left(x^2-16\times 3\right)\left((x-5)^2+2^2\right)=0 && [(a+ib)(a-ib)=a^2+b^2]\\
&\left(x^2-48\right)\left(x^2-10x+25+4\right)=0 && \text{[Simplify]}\\
&\left(x^2-48\right)\left(x^2-10x+29\right)=0 &&
\end{aligned}
$$

$$x^4 - 10x^3 + 29x^2 - 48x^2 + 480x - 1392 = 0 \quad \text{[Multiply]}$$
$$x^4 - 10x^3 - 19x^2 + 480x - 1392 = 0$$

The required equation is $x^4 - 10x^3 - 19x^2 + 480x - 1392 = 0$.

Problem 2.1.5: Form the lowest degree equation with rational coefficients one of whose roots is $3 - 4i$.

Solution: Let $x = 3 - 4i$. This is an equation with imaginary coefficient.
So, we should remove the imaginary number as explained below.

$$x = 3 - 4i \implies x - 3 = -4i$$
$$\implies (x-3)^2 = (-4i)^2 \quad \text{[Square both sides]}$$
$$\implies x^2 - 6x + 9 = 16i^2 \quad \text{[Simplify]}$$
$$\implies x^2 - 6x + 9 = 16(-1) \quad [i^2 = -1]$$
$$\implies x^2 - 6x + 9 + 16 = 0$$
$$\implies x^2 - 6x + 25 = 0$$

The required equation is $x^2 - 6x + 25 = 0$.

Problem 2.1.6: Form the lowest degree equation with rational coefficients whose roots are $4\sqrt{3}$ and $5 + 2i$.

Solution: Since $4\sqrt{3}$ is a root, its conjugate $-4\sqrt{3}$ is also a root. Since $5 + 2i$ is a root, its complex conjugate $5 - 2i$ is also a root. Hence, the required equation is

$$\left(x - 4\sqrt{3}\right)\left(x + 4\sqrt{3}\right)(x - (5+2i))(x - (5-2i)) = 0$$
$$\left(x^2 - (4\sqrt{3})^2\right)((x-5) - 2i)((x-5) + 2i) = 0 \quad [(a+b)(a-b) = a^2 - b^2]$$
$$\left(x^2 - 16 \times 3\right)\left((x-5)^2 + 2^2\right) = 0 \quad [(a+ib)(a-ib) = a^2 + b^2]$$
$$\left(x^2 - 48\right)\left(x^2 - 10x + 25 + 4\right) = 0 \quad \text{[Simplify]}$$
$$\left(x^2 - 48\right)\left(x^2 - 10x + 29\right) = 0$$
$$x^4 - 10x^3 + 29x^2 - 48x^2 + 480x - 1392 = 0 \quad \text{[Multiply]}$$
$$x^4 - 10x^3 - 19x^2 + 480x - 1392 = 0$$

The required equation is $x^4 - 10x^3 - 19x^2 + 480x - 1392 = 0$.

Problem 2.1.7: Form the lowest degree equation with rational coefficients whose roots are $4\sqrt{3}$ and $5 + 2i$.

Solution: Since $4\sqrt{3}$ is a root, its conjugate $-4\sqrt{3}$ is also a root. Since $5 + 2i$ is a root, its complex conjugate $5 - 2i$ is also a root. Hence, the required equation is

$$\left(x - 4\sqrt{3}\right)\left(x + 4\sqrt{3}\right)(x - (5+2i))(x - (5-2i)) = 0$$
$$\left(x^2 - (4\sqrt{3})^2\right)((x-5) - 2i)((x-5) + 2i) = 0 \quad [(a+b)(a-b) = a^2 - b^2]$$
$$\left(x^2 - 16 \times 3\right)\left((x-5)^2 + 2^2\right) = 0 \quad [(a+ib)(a-ib) = a^2 + b^2]$$
$$\left(x^2 - 48\right)\left(x^2 - 10x + 25 + 4\right) = 0 \quad \text{[Simplify]}$$
$$\left(x^2 - 48\right)\left(x^2 - 10x + 29\right) = 0$$

$x^4 - 10x^3 + 29x^2 - 48x^2 + 480x - 1392 = 0$ [Multiply]

$x^4 - 10x^3 - 19x^2 + 480x - 1392 = 0$

The required equation is $x^4 - 10x^3 - 19x^2 + 480x - 1392 = 0$.

2.2 Solving Equations with given irrational / imaginary roots

- Let the given equation be $f(x) = 0$.
- Using the given irrational / imaginary roots find the factor of $f(x)$ with rational coefficients, say $g(x)$.
- Divide $f(x)$ by $g(x)$ and let the quotient be $h(x)$.
- By Factor theorem, $f(x) = g(x)h(x)$.
- The remaining roots are the roots of $h(x) = 0$.
- Solve $h(x) = 0$ to find the remaining roots.

Problem 2.2.1: Solve the equation $x^5 - x^4 + 8x^2 - 9x - 15 = 0$ if $\sqrt{3}$ and $1 + 2i$ are two of its roots.

Solution: Since $\sqrt{3}$ is a root, its conjugate $-\sqrt{3}$ is also a root[Irrational roots occur in pair]. Since $1 + 2i$ is a root, its complex conjugate $1 - 2i$ is also a root[Complex roots occur in pair]. Hence we have four of its roots namely $\sqrt{3}$, $-\sqrt{3}$, $1 + 2i$ and $1 - 2i$. Therefore, $\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)(x - (1 + 2i))(x - (1 - 2i))$ is a factor of the given polynomial $x^5 - x^4 + 8x^2 - 9x - 15$.

$$\begin{aligned}
\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)(x - (1 + 2i))(x - (1 - 2i)) &= \left(x^2 - (\sqrt{3})^2\right)((x - 1) - 2i)((x - 1) + 2i) \\
&= \left(x^2 - 3\right)((x - 1) - 2i)((x - 1) + 2i) \\
&= \left(x^2 - 3\right)\left((x - 1)^2 + 2^2\right) \\
&= \left(x^2 - 3\right)\left(x^2 - 2x + 5\right) \\
&= x^4 - 2x^3 + 5x^2 - 3x^2 + 6x - 15 \\
&= x^4 - 2x^3 + 2x^2 + 6x - 15
\end{aligned}$$

Dividing the given polynomial $x^5 - x^4 + 8x^2 - 9x - 15$ by $x^4 - 2x^3 + 2x^2 + 6x - 15$ we get the remaining factor $x + 1$. Solving $x + 1 = 0$, we get the other root $x = -1$. Thus, the roots of the given equation are $\pm\sqrt{3}$, $1 \pm 2i$ and -1.

Problem 2.2.2: One root of the equation $2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81 = 0$ is $\sqrt{2} + i$. Find the remaining roots.

Solution: Since irrational roots and complex roots occur in pairs, the given equation is having roots $\sqrt{2} + i$, $\sqrt{2} - i$, $-\sqrt{2} + i$, $-\sqrt{2} - i$.

Therefore, $\left(x - (\sqrt{2} + i)\right)\left(x - (\sqrt{2} - i)\right)\left(x - (-\sqrt{2} + i)\right)\left(x - (-\sqrt{2} - i)\right)$ is a factor of the given polynomial $2x^6 - 3x^5 + 5x^4 + 6x^2 - 27x + 81$. Instead of multiplying all these factors, let us find the smallest degree equation with the above roots.

$x = \sqrt{2} + i \implies x - \sqrt{2} = i$

$$\implies \left(x-\sqrt{2}\right)^2 = (i)^2 \qquad \text{[Square both sides]}$$
$$\implies x^2 - 2\sqrt{2}x + 2 = -1 \qquad [i^2 = -1]$$
$$\implies x^2 + 3 = 2\sqrt{2}x$$
$$\implies \left(x^2+3\right)^2 = \left(2\sqrt{2}x\right)^2 \qquad \text{[Square both sides]}$$
$$\implies x^4 + 6x^2 + 9 = 8x^2$$
$$\implies x^4 - 2x^2 + 9 = 0$$

Therefore, $x^4 - 2x^2 + 9$ is a factor of the polynomial $2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81$. Dividing the given polynomial $2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81$ by $x^4 - 2x^2 + 9$ we get the remaining factor $2x^2 - 3x + 9$. Solving $2x^2 - 3x + 9 = 0$, we get the other two roots. We solve using quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad [\text{Here, } a = 2,\ b = -3,\ c = 9]$$
$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 2 \times 9}}{2 \times 2}$$
$$= \frac{3 \pm \sqrt{9 - 72}}{4}$$
$$= \frac{3 \pm \sqrt{-63}}{4}$$
$$= \frac{3 \pm \sqrt{9 \times -7}}{4}$$
$$= \frac{3 \pm i3\sqrt{7}}{4}$$
$$= \frac{3(1 \pm i\sqrt{7})}{4}$$

Thus, the roots of the given equation are $\sqrt{2}+i$, $\sqrt{2}-i$, $-\sqrt{2}+i$, $-\sqrt{2}-i$, $\frac{3(1+i\sqrt{7})}{4}$ and $\frac{3(1-i\sqrt{7})}{4}$.

Chapter 3

Relation between roots and coefficients

Consider the n^{th} degree polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, $a_0 \neq 0$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the n roots of $f(x) = 0$. Then we can write the equation as, $f(x) = a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$. Now, we have,

$$a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_n$$

$$(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) = x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \frac{a_3}{a_0}x^{n-3} + \cdots + \frac{a_n}{a_0}$$

$$x^n - \left(\sum \alpha_1\right)x^{n-1} + \left(\sum \alpha_1\alpha_2\right)x^{n-2} - \left(\sum \alpha_1\alpha_2\alpha_3\right)x^{n-3} + \cdots \cdots + (-1)^n\alpha_1\alpha_2\cdots\alpha_n = x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \frac{a_3}{a_0}x^{n-3} + \cdots + \frac{a_n}{a_0}$$

$$x^n - S_1x^{n-1} + S_2x^{n-2} - S_3x^{n-3} + \cdots + (-1)^nS_n = x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \frac{a_3}{a_0}x^{n-3} + \cdots + \frac{a_n}{a_0}$$

where

$$S_1 = \sum \alpha_1 = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n$$
= Sum of the roots

$$S_2 = \sum \alpha_1\alpha_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots$$
= Sum of the product of the roots taken two at a time

$$S_3 = \sum \alpha_1\alpha_2\alpha_3 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \cdots$$
= Sum of the product of the roots taken three at a time

$\vdots$

$$S_n = \alpha_1\alpha_2\cdots\alpha_n$$
= Product of the roots

Equating the coefficients of like terms in
$x^n - S_1x^{n-1} + S_2x^{n-2} - S_3x^{n-3} + \cdots + (-1)^nS_n = x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \frac{a_3}{a_0}x^{n-3} + \cdots + \frac{a_n}{a_0}$,
we get

$$S_1 = -\frac{a_1}{a_0}$$

$$S_2 = \frac{a_2}{a_0}$$

$$S_3 = \frac{a_3}{a_0}$$

$\vdots$

$S_n = (-1)^n \frac{a_n}{a_0}$

1. If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$S_1 = \alpha + \beta = -\frac{b}{a}$$
$$S_2 = \alpha\beta = \frac{c}{a}$$

2. If α, β and γ are the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$$S_1 = \alpha + \beta + \gamma = -\frac{b}{a}$$
$$S_2 = \alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$$
$$S_3 = \alpha\beta\gamma = -\frac{d}{a}$$

3. If α, β, γ and δ are the roots of the biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$S_1 = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$$
$$S_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$
$$S_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$$
$$S_4 = \alpha\beta\gamma\delta = \frac{e}{a}$$

3.1 Roots are in A.P.

First we solve equations whose roots are in arithmetic progression(A.P.). Also we evolve the condition for an equation to have roots in A.P.

1. A sequence of the form a, $a + d$, $a + 2d$, $\cdots$, $a + (n-1)d$, *cdots* is called an arithmetic progression. a is the first term and d is the common difference.

2. If α, β, γ are in A.P, then $2\beta = \alpha + \gamma$.

3. If the roots of a **cubic equation** are in A.P, then we take the roots as $a - d$, a, $a + d$ so that we can find a from S_1.

4. If the roots of a **biquadratic equation** are in A.P, then we take the roots as $a - 3d$, $a - d$, $a + d$, $a + 3d$ with difference $2d$ so that we can find a from S_1.

Problem 3.1.1: Show that the roots of the equation $px^3 + qx^2 + rx + s = 0$ are in arithmetic progression iff $2q^3 + 27p^2s = 9pqr$. Hence or otherwise solve $x^3 - 6x^2 + 13x - 10 = 0$.

Solution: Since the roots are in AP, let the roots be $a-d$, a and $a+d$. We have,

$$S_1 a-d+a+a+d=-\frac{q}{p}$$

$$3a=-\frac{q}{p} \qquad \cdots\cdots(1)$$

$$S_2=(a-d)a+a(a+d)+(a-d)(a+d)=\frac{r}{p}$$

$$3a^2-d^2=\frac{r}{p} \qquad \cdots\cdots(2)$$

$$S_3=(a-d)a(a+d)=-\frac{s}{p}$$

$$a(a^2-d^2)=-\frac{s}{p} \qquad \cdots\cdots(3)$$

From (1) $$a=-\frac{q}{3p} \qquad \cdots\cdots(4)$$

From (2) $$d^2=3a^2-\frac{r}{p}$$

$$\implies d^2=3\left[-\frac{q}{3p}\right]^2-\frac{r}{p}$$

$$\implies d^2=\frac{q^2}{3p^2}-\frac{r}{p} \qquad \cdots\cdots(5)$$

Substituting the values of a and d in (3), we get the condition.

(3) is $$a(a^2-d^2)=-\frac{s}{p}$$

$$\implies a^3-ad^2=-\frac{s}{p}$$

$$\implies \left(-\frac{q}{3p}\right)^3-\left(-\frac{q}{3p}\right)\left(\frac{q^2}{3p^2}-\frac{r}{p}\right)=-\frac{s}{p}$$

$$\implies -\frac{q^3}{27p^3}+\frac{q^3}{9p^3}-\frac{qr}{3p^2}=-\frac{s}{p}$$

$$\implies \frac{-q^3+3q^3-9pqr}{27p^3}=-\frac{s}{p}$$

$$\implies -\frac{2q^3-9pqr}{27p^3}=-\frac{s}{p}$$

$$\implies p(2q^3-9pqr)=-27p^3s$$

$$\implies 2q^3-9pqr=-27p^2s$$

$$\implies 2q^3+27p^2s=9pqr$$

This is the required condition.

Next let us solve $x^3-6x^2+13x-10=0$. $p=1$, $q=-6$, $r=13$, $s=-10$. Let us check the condition $2q^3+27p^2s=9pqr$.

$$\begin{aligned}2q^3+27p^2s&=2\times(-6)^3+27\times 1^2\times(-10)\\&=2\times -216-270\\&=-432-270\end{aligned}$$

$$= -702$$
$$9pqr = 9 \times 1 \times (-6) \times 13$$
$$= -702$$
$$\implies \quad 2q^3 + 27p^2s = -9pqr$$

Hence the roots are in A.P and they are $a-d$, a and $a+d$.

From (4)
$$a = -\frac{q}{3p} = -\frac{-6}{3} = 2$$

From (5)
$$d^2 = \frac{q^2}{3p^2} - \frac{r}{p} = \frac{36}{3} - \frac{13}{1} = 12 - 13 = -1$$
$$d = \pm i \qquad [\sqrt{-1} = \pm i]$$

Let us take $d = i$. [Note that even if we take $d = -i$, we will get the same answer.] The roots are $2-i$, 2, $2+i$.

Problem 3.1.2: Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$ given that the roots are in arithmetic progression.

Solution: Since the roots are in AP, let the roots be $a-d$, a and $a+d$. We have,

$$a - d + a + a + d = 6 \qquad \left[S_1 = -\frac{-24}{4}\right]$$
$$3a = 6$$
$$a = 2 \qquad \cdots\cdots(1)$$
$$(a-d)a + a(a+d) + (a-d)(a+d) = \frac{23}{4}$$
$$3a^2 - d^2 = \frac{23}{4} \qquad \cdots\cdots(2)$$
$$(a-d)a(a+d) = -\frac{18}{4}$$
$$a(a^2 - d^2) = -\frac{9}{2} \qquad \cdots\cdots(3)$$

From (2)
$$d^2 = 3a^2 - \frac{23}{4}$$
$$\implies \quad d^2 = 3 \times 4 - \frac{23}{4} \qquad [a = 2]$$
$$\implies \quad d^2 = \frac{48 - 23}{4}$$
$$\implies \quad d^2 = \frac{25}{4}$$

$$\implies \qquad d = \pm\frac{5}{2}$$

Let us take $d = \frac{5}{2}$. Therefore, the roots are $2 - \frac{5}{2}$, 2, $2 + \frac{5}{2}$. That is, $-\frac{1}{2}$, 2, $\frac{9}{2}$.

Problem 3.1.3: Solve the equation $x^4 + 4x^3 - 34x^2 - 76x + 105 = 0$ given that the roots are in arithmetic progression.

Solution: Since the roots are in AP, let the roots be $a - 3d$, $a - d$, $a + d$ and $a + 3d$. We have,

$$S_1 = -\frac{b}{a} \implies \quad a - 3d + a - d + a + d + a + 3d = -4$$
$$4a = -4$$
$$a = -1 \qquad \cdots\cdots(1)$$
$$S_4 = \frac{e}{a} \implies \quad (a-3d)(a-d)(a+d)(a+3d) = 105$$
$$(a^2 - d^2)(a^2 - 9d^2) = 105$$
$$\implies \quad (1 - d^2)(1 - 9d^2) = 105 \qquad [\because a = -1]$$
$$\implies \quad 1 - 10d^2 + 9d^4 = 105$$
$$\implies \quad 9d^4 - 10d^2 - 104 = 0 \qquad \cdots\cdots(2)$$

(2) is a quadratic equation in d^2. Solving (2) we have

$$d^2 = \frac{-(-10) \pm \sqrt{(-10)^2 - 4 \times 9 \times -104}}{2 \times 9}$$
$$= \frac{10 \pm \sqrt{100 + 3744}}{18}$$
$$= \frac{10 \pm \sqrt{3844}}{18}$$
$$= \frac{10 \pm 62}{18}$$
$$= \frac{72}{18}, \frac{-52}{18}$$
$$\implies \quad d^2 = \frac{72}{18} \qquad [\text{Since } d^2 \text{ is non negative}]$$
$$= 4$$
$$\implies \quad d = \pm 2$$

Let us take $d = 2$.

The roots are $a - 3d, \quad a - d, \quad a + d, \quad a + 3d$

$-1 - 6, \quad -1 - 2, \quad -1 + 2, \quad -1 + 6$

$-7, \quad -3, \quad 1, \quad 5$

Problem 3.1.4: Find the value of k for which the roots of the equation $x^3 - 9x^2 + 25x - k = 0$ are in arithmetic progression.

Solution: Since the roots are in AP, let the roots be $a - d$, a and $a + d$. We have,

$$a - d + a + a + d = 9$$

$$
\begin{aligned}
3a &= 9 \\
a &= 3 && \cdots\cdots(1) \\
(a-d)a + a(a+d) + (a-d)(a+d) &= 25 \\
3a^2 - d^2 &= 25 \\
\text{From (2)} \quad d^2 &= 3a^2 - 25 \\
\implies d^2 &= 3 \times 3^2 - 25 && [a = 3] \\
\implies d^2 &= 27 - 25 \\
\implies d^2 &= 2 && \cdots\cdots(2) \\
(a-d)a(a+d) &= k && \left[S_3 = -\frac{-k}{1}\right] \\
a(a^2 - d^2) &= k && \cdots\cdots(3) \\
\implies k &= a(a^2 - d^2) \\
&= 3(3^2 - 2) && \text{[From (1) and (2)]} \\
&= 21
\end{aligned}
$$

3.2 Roots are in G.P.

Problem 3.2.1: Show that the roots of the equation $px^3 + qx^2 + rx + s = 0$ are in geometric progression iff $r^3p = q^3s$.

Solution: Since the roots are in G.P, let the roots be $\frac{a}{k}$, a and ak. We have,

$$
\begin{aligned}
\frac{a}{k} + a + ak &= -\frac{q}{p} && [S_1] \\
a\left(\frac{1}{k} + 1 + k\right) &= -\frac{q}{p} && \cdots\cdots(1) \\
\frac{a}{k}\cdot a + a\cdot ak + \frac{a}{k}\cdot ak &= \frac{r}{p} && [S_2] \\
\frac{a^2}{k} + a^2k + a^2 &= \frac{r}{p} \\
a^2\left(\frac{1}{k} + k + 1\right) &= \frac{r}{p} && \cdots\cdots(2) \\
\frac{a}{k}\cdot a \cdot ak &= -\frac{s}{p} && [S_3] \\
a^3 &= -\frac{s}{p} && \cdots\cdots(3) \\
\frac{(2)}{(1)} \implies a &= \frac{r}{p}\cdot -\frac{p}{q} \\
\implies a &= -\frac{r}{q} && \cdots\cdots(4) \\
\text{From (3)} \quad a^3 &= -\frac{s}{p} \\
\implies \left(-\frac{r}{q}\right)^3 &= -\frac{s}{p}
\end{aligned}
$$

$$\implies \qquad -\frac{r^3}{q^3} = -\frac{s}{p}$$

$$\implies \qquad r^3 p = q^3 s$$

Problem 3.2.2: Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ if the roots are in geometric progression.

Solution: Since the roots are in G.P, let the roots be $\frac{a}{r}$, a and ar. We have,

$$\frac{a}{r} + a + ar = -\frac{42}{27} \qquad [S_1]$$

$$a\left(\frac{1}{r} + 1 + r\right) = -\frac{14}{9} \qquad \cdots\cdots(1)$$

$$\frac{a}{r}\cdot a + a\cdot ar + \frac{a}{r}\cdot ar = \frac{-28}{27} \qquad [S_2]$$

$$\frac{a^2}{r} + a^2 r + a^2 = -\frac{28}{27}$$

$$a^2\left(\frac{1}{r} + r + 1\right) = -\frac{28}{27} \qquad \cdots\cdots(2)$$

$$\frac{a}{r}\cdot a\cdot ar = -\frac{-8}{27} \qquad [S_3]$$

$$a^3 = \frac{8}{27} \qquad \cdots\cdots(3)$$

From (3)
$$a = \left[\frac{8}{27}\right]^{\frac{1}{3}}$$

$$\implies \qquad a = \frac{2}{3} \qquad \cdots\cdots(4)$$

Substitute $a = \frac{2}{3}$ in equation (1).

From (1)
$$a\left(\frac{1}{r} + 1 + r\right) = -\frac{14}{9}$$

$$\implies \qquad \frac{2}{3}\left(\frac{1}{r} + 1 + r\right) = -\frac{14}{9}$$

$$\implies \qquad \frac{1 + r + r^2}{r} = -\frac{14}{9}\cdot\frac{3}{2}$$

$$\implies \qquad \frac{1 + r + r^2}{r} = -\frac{7}{3}$$

$$\implies \qquad 1 + r + r^2 = -\frac{7r}{3}$$

$$\implies \qquad 3 + 3r + 3r^2 = -7r$$

$$\implies \qquad 3r^2 + 10r + 3 = 0$$

$$\implies \qquad 3r^2 + 9r + r + 3 = 0$$

$$\implies \qquad 3r(r+3) + 1(r+3) = 0$$

$$\implies \qquad (3r+1)(r+3) = 0$$

$$\implies \qquad r = -3,\ -\frac{1}{3}$$

Let us take $r = -3$.

The roots are $\frac{a}{r}, \quad a, \quad ar$

$$\frac{\frac{2}{3}}{-3}, \quad \frac{2}{3}, \quad \frac{2}{3}\cdot(-3)$$

$$-\frac{2}{9}, \quad \frac{2}{3}, \quad -2$$

Note: If we take $-\frac{1}{3}$ then also we get the same answer.

Problem 3.2.3: Solve $3x^4 - 40x^3 + 130x^2 - 120x + 27 = 0$ whose roots are in geometric progression.

Solution: Let the roots be α, β, γ and δ. Since the roots are in G.P, We have $\frac{\alpha}{\beta} = \frac{\beta}{\gamma} = \frac{\gamma}{\delta}$.

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \implies \alpha\delta = \beta\gamma \cdots\cdots (1)$$

$$S_4 = \frac{e}{a} \implies \alpha\beta\gamma\delta = \frac{2}{3}$$

$$\implies (\alpha\delta)(\beta\gamma) = \frac{27}{3} \quad [\text{Since } \alpha\delta = \beta\gamma$$

$$\implies (\alpha\delta)(\alpha\delta) = 9 \quad [\text{Since } \alpha\delta = \beta\gamma$$

$$\implies (\alpha\delta)^2 = 9$$

$$\implies \alpha\delta = 3$$

$$\implies \alpha\delta = \beta\gamma = 3 \quad \cdots\cdots (2)$$

We will find $\alpha + \delta$ and $\beta + \gamma$ first and then we find the roots. We have,

$$3(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \equiv 3x^4 - 40x^3 + 130x^2 - 120x + 27$$

$$\implies 3(x-\alpha)(x-\delta)(x-\beta)(x-\gamma) \equiv 3x^4 - 40x^3 + 130x^2 - 120x + 27$$

$$\implies 3\left[x^2 - (\alpha+\delta)x + \alpha\delta\right]\left[x^2 - (\beta+\gamma)x + \beta\gamma\right] \equiv 3x^4 - 40x^3 + 130x^2 - 120x + 27$$

Let $\alpha + \delta = p$ and $\beta\gamma = q$. Also we have $\alpha\delta = \beta\gamma = 3$.

$$\implies 3\left[x^2 - px + 3\right]\left[x^2 - qx + 3\right] \equiv 3x^4 - 40x^3 + 130x^2 - 120x + 27$$

$$\implies 3\left[x^4 - (p+q)x^3 + (pq+6)x^2 - (p+q)x + 9\right] \equiv 3x^4 - 40x^3 + 130x^2 - 120x + 27$$

Equating coefficients of x^3 and x^2 separately we get

$$3(p+q) = 40 \quad [\text{coefficients of } x^3]$$

$$\implies (p+q) = \frac{40}{3} \quad \cdots\cdots (3)$$

$$3(pq+6) = 130 \quad [\text{coefficients of } x^2](3)$$

$$\implies (pq+6) = \frac{130}{3}$$

$$\implies pq = \frac{130}{3} - 6$$

$$\implies pq = \frac{112}{3} \quad \cdots\cdots (4)$$

Now, $(p-q)^2 = (p+q)^2 - 4pq$

$$\implies \quad (p-q)^2 = \left(\frac{40}{3}\right)^2 - 4\frac{112}{3} \qquad \text{[From (3) and (4)]}$$

$$\implies \quad (p-q)^2 = \frac{1600}{9} - \frac{448}{3}$$

$$\implies \quad (p-q)^2 = \frac{1600-1344}{9}$$

$$\implies \quad (p-q)^2 = \frac{256}{9}$$

$$\implies \quad (p-q) = \frac{16}{3} \qquad \cdots\cdots(5)$$

Solve (3) and (5) to get p and q.

$$(3)+(5) \implies \quad 2p = \frac{40}{3} + \frac{16}{3} = \frac{56}{3}$$

$$\implies \quad p = \frac{56}{6}$$

$$\implies \quad \alpha + \delta = \frac{28}{3} \qquad [p = \alpha + \delta] \qquad \cdots\cdots(6)$$

$$(3)-(5) \implies \quad 2q = \frac{40}{3} - \frac{16}{3} = \frac{24}{3} = 8$$

$$\implies \quad q = 4$$

$$\implies \quad \beta + \gamma = 4 \qquad [q = \beta + \gamma] \qquad \cdots\cdots(7)$$

From (6) and (2) we can find α and δ. Similarly from (7) and (2) we can find β and γ.

From (6) and (2),
$$\begin{aligned}(\alpha-\delta)^2 &= (\alpha+\delta)^2 - 4\alpha\delta \\ &= \left(\frac{28}{3}\right)^2 - 4\cdot 3 \\ &= \frac{784}{9} - 12 \\ &= \frac{676}{9}\end{aligned}$$

$$\implies \quad \alpha - \delta = \frac{26}{3} \qquad \cdots\cdots(8)$$

$$(6)+(8) \implies \quad 2\alpha = \frac{54}{3}$$

$$\implies \quad \alpha = 9$$

$$(6)\text{ - }(8) \implies \quad 2\delta = \frac{2}{3}$$

$$\implies \quad \delta = \frac{1}{3}$$

From (7) and (2),
$$\begin{aligned}(\beta-\gamma)^2 &= (\beta+\gamma)^2 - 4\beta\gamma \\ &= 4^2 - 4\cdot 3 \\ &= 4\end{aligned}$$

$$\implies \quad \beta - \gamma = 2 \qquad \cdots\cdots(9)$$

$$(7)+(9) \implies \quad 2\beta = 6$$

$$\implies \quad \beta = 3$$

$$\text{(7) - (9)} \implies \quad 2\gamma = 2$$
$$\implies \quad \gamma = 1$$

The roots are 9, 3, 1 and $\frac{1}{3}$.

3.3 Roots are in H.P.

If α, β and γ are in harmonic progression, then their reciprocals $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ are in harmonic progression. Hence we have, $\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma}$.

Problem 3.3.1: Solve $6x^3 - 11x^2 + 6x - 1 = 0$ where roots are in harmonic progression.

Solution: Let the roots be α, b and γ. Given that the roots are in harmonic progression. So, we have

$$\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma}$$
$$\frac{2}{\beta} = \frac{\gamma + \alpha}{\alpha\gamma} \qquad \text{[Simplify]}$$
$$2\alpha\gamma = \alpha\beta + \beta\gamma \qquad \cdots\cdots(1)$$

Further we use the relations between roots and coefficients and have the following equations.

$$\alpha + \beta + \gamma = \frac{11}{6} \qquad \cdots\cdots(2) \left[S_1 = -\frac{b}{a} = -\frac{-11}{6} = \frac{11}{6}\right]$$
$$\alpha\beta + \beta\gamma + \alpha\gamma = 1 \qquad \cdots\cdots(3) \left[S_2 = \frac{c}{a} = \frac{6}{6} = 1\right]$$
$$\alpha\beta\gamma = \frac{1}{6} \qquad \cdots\cdots(4) \left[S_3 = -\frac{d}{a} = -\frac{-1}{6} = \frac{1}{6}\right]$$

$$\text{From (1) and (3):} \quad 3\alpha\gamma = 1 \qquad [\alpha\beta + \beta\gamma = 2\alpha\gamma]$$
$$\implies \quad \alpha\gamma = \frac{1}{3} \qquad \cdots\cdots(5)$$
$$\text{Substitute in (4):} \quad \frac{1}{3}\beta = \frac{1}{6}$$
$$\implies \quad \beta = \frac{1}{2}$$
$$\text{Put } \beta = \frac{1}{2} \text{ in (2)} \quad \alpha + \frac{1}{2} + \gamma = \frac{11}{6}$$
$$\implies \quad \alpha + \gamma = \frac{11}{6} - \frac{1}{2}$$
$$\implies \quad \alpha + \gamma = \frac{11-3}{6} = \frac{8}{6}$$
$$\implies \quad \alpha + \gamma = \frac{4}{3} \qquad \cdots\cdots(6)$$
$$\text{Now,} \quad (\alpha - \gamma)^2 = (\alpha + \gamma)^2 - 4\alpha\gamma$$
$$\implies \quad (\alpha - \gamma)^2 = \left(\frac{4}{3}\right)^2 - 4 \cdot \frac{1}{3} \qquad \text{[Using (5) and (6)]}$$
$$\implies \quad (\alpha - \gamma)^2 = \frac{16}{9} - \frac{4}{3}$$

$$\implies \qquad (\alpha-\gamma)^2 = \frac{4}{9}$$

$$\implies \qquad \alpha-\gamma = \frac{2}{3} \qquad \cdots\cdots(7)$$

$$(6)+(7) \implies \qquad 2\alpha = \frac{4}{3}+\frac{2}{3} = 2$$

$$\implies \qquad \alpha = 1$$

$$\text{From (6)} \qquad \gamma = \frac{4}{3}-\alpha$$

$$\implies \qquad \gamma = \frac{4}{3}-1$$

$$\implies \qquad \gamma = \frac{1}{3}$$

The roots are 1, $\frac{1}{2}$ and $\frac{1}{3}$.

3.4 Miscellaneous conditions

Problem 3.4.1: The product of two roots of $x^4+px^3+qx^2+rx+s=0$ is equal to the product of the other two. Show that, $r^2=p^2s$.

Solution: Let the roots be α, β, γ and δ. Since the product of two roots is equal to the product of the other two, we have

$$\alpha\beta = \gamma\delta \qquad \cdots\cdots(1)$$

$$\text{We have,} \qquad \alpha\beta\gamma\delta = s$$

$$\implies \qquad (\alpha\beta)^2 = s$$

$$\implies \qquad \alpha\beta = \sqrt{s} \qquad \cdots\cdots(2)$$

$$S1 = -\frac{b}{a} \implies \qquad \alpha+\beta+\gamma+\delta = -p \qquad \cdots\cdots(3)$$

$$S3 = -\frac{d}{a} \implies \qquad \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -r \qquad \cdots\cdots(4)$$

$$\implies \qquad \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) = -r$$

$$\implies \qquad \alpha\beta(\gamma+\delta)+\alpha\beta(\alpha+\beta) = -r \qquad [\text{Since } \alpha\beta=\gamma\delta]$$

$$\implies \qquad \alpha\beta(\alpha+\beta+\gamma+\delta) = -r$$

$$\implies \qquad \sqrt{s}(-p) = -r \qquad [\text{From (2) and (3)}]$$

$$\implies \qquad \left[\sqrt{s}(-p)\right]^2 = (-r)^2 \qquad [\text{Squaring both sides}]$$

$$\implies \qquad sp^2 = r^2$$

$$\implies \qquad r^2 = ps$$

Problem 3.4.2: If the sum of two roots of $x^4+px^3+qx^2+rx+s=0$ is equal to the sum of the other two, show that, $p^3+8r=4pq$.

Solution: Let the roots be α, β, γ and δ. Since the sum of two roots is equal to the sum of the other two, we have

$$\alpha+\beta = \gamma+\delta \qquad \cdots\cdots(1)$$

We have,

$$S1 = -\frac{b}{a} \implies \alpha + \beta + \gamma + \delta = -p \qquad \cdots\cdots(2)$$

$$S2 = \frac{d}{a} \implies \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q \qquad \cdots\cdots(3)$$

$$S3 = -\frac{d}{a} \implies \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r \qquad \cdots\cdots(4)$$

$$S4 = \frac{e}{a} \implies \alpha\beta\gamma\delta = s \qquad \cdots\cdots(5)$$

From (1) and (2), $2(\alpha + \beta) = -p$

$$\implies \alpha + \beta = -\frac{p}{2} \qquad \cdots\cdots(6)$$

From (3), $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$

$$\implies \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \alpha\beta + \gamma\delta = q$$

$$\implies \alpha(\gamma + \delta) + \beta(\gamma + \delta) + \alpha\beta + \gamma\delta = q$$

$$\implies (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = q$$

$$\implies (\alpha + \beta)^2 + \alpha\beta + \gamma\delta = q$$

$$\implies \frac{p^2}{4} + \alpha\beta + \gamma\delta = q \qquad \cdots\cdots(7)$$

From (4), $\alpha\beta(\gamma + \delta) + \gamma\delta + \gamma\delta(\alpha + \beta) = -r$

$$\implies \alpha\beta(\alpha + \beta) + \gamma\delta + \gamma\delta(\alpha + \beta) = -r$$

$$\implies (\alpha + \beta)(\alpha\beta + \gamma\delta) = -r$$

$$\implies -\frac{p}{2}(\alpha\beta + \gamma\delta) = -r \qquad \cdots\cdots(8)$$

Eliminating $\alpha\beta + \gamma\delta$, we will get the condition.

From (7), $$\alpha\beta + \gamma\delta = q - \frac{p^2}{4} \qquad \cdots\cdots(9)$$

From (8), $$\alpha\beta + \gamma\delta = \frac{2r}{p} \qquad \cdots\cdots(10)$$

From (9) and (10), $$= q - \frac{p^2}{4} = \frac{2r}{p}$$

$$\implies 4pq - p^3 = = 8r$$

$$\implies p^3 + 8r = = 4pq$$

Problem 3.4.3: Solve $4x^3 - 12x^2 - 15x - 4 = 0$ given that it has a double root.

Solution: Let the roots be α, α and β, since it has two equal roots. From the relations between roots and coefficients, we have

$$\alpha + \alpha + \beta = -\frac{-12}{4} \qquad [S_1]$$

$$\implies 2\alpha + \beta = 3 \qquad \cdots\cdots(1)$$

$$\alpha \cdot \alpha + \alpha\beta + \alpha\beta = \frac{-15}{4} \qquad [S_2]$$

$$\implies \alpha^2 + 2\alpha\beta = 3 \qquad \cdots\cdots(2)$$

$$\alpha^2\beta = 1 \qquad \cdots\cdots(3)$$

From (1), $$\beta = 3 - 2\alpha$$

Substituting in (2) we get

$$\alpha^2 + 2\alpha(3-2\alpha) = -\frac{15}{4}$$

$$\implies \alpha^2 + 6\alpha - 4\alpha^2 = -\frac{15}{4}$$

$$\implies 3\alpha^2 - 6\alpha - \frac{15}{4} = 0$$

$$\implies 12\alpha^2 - 24\alpha - 15 = 0$$

$$\implies 12\alpha^2 - 30\alpha + 6a\alpha - 15 = 0$$

$$\implies (6\alpha+3)(2\alpha-5) = 0$$

$$\implies \alpha = -\frac{1}{2}, \quad \frac{5}{2}$$

Note that we solve (1) and (2) and got this value. We find the value of β using either (1) or (2) only. Finally we check whether the pair satisfies equation (3) also. Using equation (1) we find β.

$$\alpha = -\frac{1}{2} \implies \beta = 4 \qquad \cdots\cdots(4)$$

$$\alpha = \frac{5}{2} \implies \beta = -2 \qquad \cdots\cdots(5)$$

The roots must satisfy (3) also. Let us check with the pair $\alpha = -\frac{1}{2}$, $\beta = 4$.

$$\alpha^2\beta = \left(-\frac{1}{2}\right)^2 \times 4$$

$$= 1$$

Since (3) is also satisfied, the roots are $-\frac{1}{2}$, $-\frac{1}{2}$, 4.

Problem 3.4.4: Solve the equation $6x^4 - 3x^3 + 8x^2 - x + 2 = 0$ given that two of its roots are equal in magnitude but opposite in sign.

(OR)

Solve the equation $6x^4 - 3x^3 + 8x^2 - x + 2 = 0$ given that it has a pair of roots whose sum is zero.

Solution: Since two of its roots are equal in magnitude but opposite in sign, let us take the roots as α, $-\alpha$, β and γ, since it has two equal roots. From the relations between roots and coefficients, we have

$$\alpha - \alpha + \beta + \gamma = -\frac{-3}{6} \quad [S_1]$$

$$\implies \beta + \gamma = \frac{1}{2} \qquad \cdots\cdots(1)$$

$$\alpha(-\alpha) + \alpha\beta + \alpha\gamma + (-\alpha)\beta + (-\alpha)\gamma + \beta\gamma = \frac{8}{6} \quad [S_2]$$

$$\implies -\alpha^2 + \alpha\beta + \alpha\gamma - \alpha\beta + -\alpha\gamma + \beta\gamma = \frac{4}{3}$$

$$\implies \qquad -\alpha^2+\beta\gamma=\frac{4}{3} \qquad \cdots\cdots(2)$$

$$\alpha(-\alpha)\beta+\alpha(-\alpha)\gamma+\alpha\beta\gamma+(-\alpha)\beta\gamma=-\frac{-1}{6} \quad [S_3]$$

$$\implies \qquad -\alpha^2b-\alpha^2\gamma+\alpha\beta\gamma-\alpha\beta\gamma=\frac{1}{6}$$

$$\implies \qquad -\alpha^2(\beta+\gamma)=\frac{1}{6} \qquad \cdots\cdots(3)$$

$$\alpha(-\alpha)\beta\gamma=-\frac{2}{6} \quad [S_4]$$

$$\implies \qquad -\alpha^2\beta\gamma=\frac{1}{3} \qquad \cdots\cdots(4)$$

From (3) and (1),

$$-\frac{\alpha^2}{2}=\frac{1}{6}$$

$$\implies \qquad -\alpha^2=\frac{1}{3}$$

$$\implies \qquad \alpha^2=-\frac{1}{3}$$

$$\implies \qquad \alpha=\sqrt{-\frac{1}{3}}$$

$$\implies \qquad \alpha=\frac{i}{\sqrt{3}},\ -\frac{i}{\sqrt{3}} \qquad \cdots\cdots(5)$$

Substituting α in (4)

$$\frac{1}{3}\beta\gamma=\frac{1}{3}$$

$$\implies \qquad \beta\gamma=1 \qquad \cdots\cdots(6)$$

$$(\beta-\gamma)^2=(\beta+\gamma)^2-4\alpha\gamma$$

$$\implies \qquad =\frac{1}{4}-4\times 1 \quad \text{[From (1) and (6)]}$$

$$\implies \qquad =\frac{-15}{4}$$

$$\beta-\gamma=\sqrt{-\frac{15}{4}}$$

$$\implies \qquad \beta-\gamma=i\frac{\sqrt{15}}{2} \qquad \cdots\cdots(7)$$

Solving (1) and (7), we get

$$\beta=\frac{1+i\sqrt{15}}{4}$$

$$\gamma=\frac{1-i\sqrt{15}}{4}$$

Hence, the roots are $\frac{i}{\sqrt{3}}$, $-\frac{i}{\sqrt{3}}$, $\frac{1+i\sqrt{15}}{4}$ and $\frac{1-i\sqrt{15}}{4}$.

Problem 3.4.5: Find the condition that the equation $ax^4+4bx^3+6x^2+4dx+e=0$ may have two pair of equal roots.

Solution: Let the roots be α, α, β and β. From the relations between roots and coefficients, we have

$$\alpha+\alpha+\beta+\beta = -\frac{4b}{a} \quad [S_1]$$

$$\Longrightarrow \quad \alpha+\beta = -\frac{2b}{a} \qquad \cdots\cdots(1)$$

$$\alpha^2+4\alpha\beta+\beta^2 = \frac{6c}{a} \quad [S_2]$$

$$\Longrightarrow \quad (\alpha+\beta)^2+2\alpha\beta = \frac{6c}{a}$$

$$\Longrightarrow \quad 2\alpha\beta = \frac{6c}{a}-(\alpha+\beta)^2$$

$$\Longrightarrow \quad 2\alpha\beta = \frac{6c}{a}-\frac{4b^2}{a^2} \quad [\text{Using } (1)]$$

$$\Longrightarrow \quad 2\alpha\beta = \frac{6ac-4b^2}{a^2} \qquad \cdots\cdots(2)$$

$$2\alpha^2\beta+2\alpha\beta^2 = -\frac{4d}{a} \quad [S_3]$$

$$\Longrightarrow \quad 2\alpha\beta(\alpha+\beta) = -\frac{4d}{a} \qquad \cdots\cdots(3)$$

$$\alpha^2\beta^2 = \frac{e}{a} \quad [S_4]$$

$$\Longrightarrow \quad (\alpha\beta)^2 = \frac{e}{a} \qquad \cdots\cdots(4)$$

$(3) \div (1) \implies$

$$2\alpha\beta = -\frac{4d}{a} \times -\frac{a}{2b}$$

$$\alpha\beta = \frac{d}{b} \qquad \cdots\cdots(5)$$

From (2) and (5)

$$\frac{6ac-4b^2}{a^2} = 2\times\frac{d}{b}$$

$$\Longrightarrow \quad b(6ac-4b^2) = 2da^2$$

$$\Longrightarrow \quad 6abc-4b^3 = 2a^2d$$

$$\Longrightarrow \quad 3abc-2b^3 = a^2d \qquad \cdots\cdots(*)$$

From (4) and (5)

$$\left(\frac{d}{b}\right)^2 = \frac{e}{a}$$

$$\Longrightarrow \quad \frac{d^2}{b^2} = \frac{e}{a}$$

$$\Longrightarrow \quad ad^2-eb^2 = 0 \qquad \cdots\cdots(**)$$

Hence, the conditions are $3abc-2b^3 = a^2d$ and $ad^2-eb^2 = 0$.

Chapter 4

Symmetric functions of the roots

4.1 Symmetric functions

Definition 4.1.1: A function of roots of a polynomial equation is said to be symmetric if the function remains unchanged when the roots are interchanged.

For example, if α, β and γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$, then

1. $\alpha + \beta + \gamma$ is a symmetric function and is denoted by $\sum \alpha$.
2. $\alpha^2 + \beta^2 + \gamma^2$ is a symmetric function and is denoted by $\sum \alpha^2$.
3. $\alpha\beta + \beta\gamma + \gamma\alpha$ is a symmetric function and is denoted by $\sum \alpha\beta$.
4. $\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha$ is not a symmetric function. Let us check why?...
    - interchange α and β.
    - the function becomes $\beta^2\alpha + \alpha^2\gamma + \gamma^2\beta$.
    - $\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha \neq \beta^2\alpha + \alpha^2\gamma + \gamma^2\beta$.
    - Since the function changes, $\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha$ is not a symmetric function.
5. $\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$ is a symmetric function and is denoted by $\sum \alpha^2\beta$. Let us verify...
    - interchange α and β.
    - the function becomes $\beta^2\alpha + \beta^2\gamma + \alpha^2\beta + \alpha^2\gamma + \gamma^2\beta + \gamma^2\alpha$
    - the function remains unchanged. Similarly we can verify by interchanging any two roots.

If α, β, γ and δ are the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then the following are some of the symmetric functions of the roots.

1. $\sum \alpha = \alpha + \beta + \gamma + \delta$.
2. $\sum \frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$.
3. $\sum \alpha^2\beta = \alpha^2\beta + \alpha^2\gamma + \alpha^2\delta + \beta^2\alpha + \beta^2\gamma + \beta^2\delta + \gamma^2\alpha + \gamma^2\beta + \gamma^2\delta + \delta^2\alpha + \delta^2\beta + \delta^2\gamma$.
4. $\sum \alpha^2\beta\gamma = \alpha^2\beta\gamma + \alpha^2\beta\delta + \alpha^2\gamma\delta + \beta^2\alpha\gamma + \beta^2\alpha\delta + \beta^2\gamma\delta + \gamma^2\alpha\beta + \gamma^2\alpha\delta + \gamma^2\beta\delta + \delta^2\alpha\beta + \delta^2\alpha\gamma + \delta^2\beta\gamma$.

4.2 Basic symmetric functions

Recall that if a function involving all the roots of a polynomial equation is unaltered if any two of the roots are interchanged then it is called a symmetric function.

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of the equation $f(x)x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0$. We have,

$$\begin{aligned} S_1 &= \sum \alpha_1 && = -p_1 \\ S_2 &= \sum \alpha_1\alpha_2 && = p_2 \\ S_3 &= \sum \alpha_1\alpha_2\alpha_3 && = -p_3 \\ &\vdots && \vdots \\ S_n &= \alpha_1\alpha_2\cdots\alpha_n && = (-)^n p_n \end{aligned}$$

These are the basic symmetric functions of the roots. Using these relations, we can evaluate any symmetric function of the roots without knowing the values of the roots.

4.2.1 Some important Symmetric functions

Some important identities

In this section we listed some important symmetric functions.

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \\ &= \left(\sum \alpha\right)^2 - 2\sum \alpha\beta \\ \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 &= (a\beta + \beta\gamma + \gamma\alpha)^2 - 2\left(\alpha\beta^2\gamma + \beta\gamma^2\alpha + \alpha^2\beta\gamma\right) \\ &= (a\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \\ &= \left(\sum \alpha\beta\right)^2 - 2\alpha\beta\gamma\left(\sum \alpha\right) \end{aligned}$$

Symmetric functions of roots of a cubic equation

Let α, β, γ be the roots of $x^3+px^2+qx+r = 0$. Then we have the elementary symmetric functions

$$\begin{aligned} \sum \alpha &= \alpha + \beta + \gamma = -p \\ \sum \alpha\beta &= \alpha\beta + \beta\gamma + \alpha\gamma = q \\ \sum \alpha\beta\gamma &= \alpha\beta\gamma = -r \end{aligned}$$

Any other symmetric functions can be written in terms of these functions. Some functions are the consequences of well known formulas and some require logical thinking.

1. $\sum \alpha^2$

$$\begin{aligned} \sum \alpha^2 &= \alpha^2 + \beta^2 + \gamma^2 \\ &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \end{aligned}$$

$$= \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$$

2. $\sum \alpha^2\beta^2$

$$\begin{aligned}\sum \alpha^2\beta^2 &= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 \\ &= (a\beta + \beta\gamma + \gamma\alpha)^2 - 2\left(\alpha\beta^2\gamma + \beta\gamma^2\alpha + \alpha^2\beta\gamma\right) \\ &= (a\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \\ &= \left(\sum \alpha\beta\right)^2 - 2\alpha\beta\gamma\left(\sum \alpha\right)\end{aligned}$$

3. $\sum \alpha^2\beta$

Each term contains two factors with different degree.

α^2 Three choices α^2 or β^2 or γ^2	β Two choices from the remaining

The first factor can be α^2 or β^2 or γ^2 and hence there are 3 ways to fill the first factor. After fixing the first one we can write the second factor with any one of the remaining roots. (*If we take the same root used in the first factor then we get the cube α^3 or β^3 or γ^3*) Hence by using combination, thre are $3 \times 3 = 6$ terms in the function.

$$\sum \alpha^2\beta = \alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$$

This is a part of the product $(\alpha + \beta + \gamma)(a\beta + \alpha\gamma + \beta\gamma)$

$$\left(\sum \alpha\right)\left(\sum \alpha\beta\right) = (\alpha + \beta + \gamma)(a\beta + \alpha\gamma + \beta\gamma)$$

Note that this product has 9 terms, out of which 6 terms are in our required function and the remaining three are $\alpha\beta\gamma$.

$$\left(\sum \alpha\right)\left(\sum \alpha\beta\right) = \left(\sum \alpha^2\beta\right) + 3\alpha\beta\gamma$$

$$\sum \alpha^2\beta = \left(\sum \alpha\right)\left(\sum \alpha\beta\right) - 3\alpha\beta\gamma$$

4. $\sum \alpha^3$

$$\sum \alpha^3 = \alpha^3\beta + \beta^3 + \gamma^3$$

This is a part of the product $(\sum \alpha)(\sum \alpha^2)$

$$\begin{aligned}\left(\sum \alpha\right)\left(\sum \alpha^2\right) &= (\alpha + \beta + \gamma)(a^2 + +\beta^2 + \gamma^2) \\ &= \sum \alpha^3 + \sum \alpha^2\beta\end{aligned}$$

$$\sum \alpha^3 = \left(\sum \alpha\right)\left(\sum \alpha^2\right) - \sum \alpha^2\beta$$

5. $\sum \alpha^2\beta\gamma$

$$\begin{aligned}\sum \alpha^2\beta\gamma &= \alpha^2\beta\gamma + \beta^2\alpha\gamma + \gamma^2\alpha\beta \\ &= \alpha\beta\gamma(\alpha+\beta+\gamma) \\ &= \alpha\beta\gamma\left(\sum \alpha\right)\end{aligned}$$

Symmetric functions of roots of a biquadratic equation

Let $\alpha, \beta, \gamma, \delta$ be the roots of $x^4 + px^3 + qx^2 + rx + s = 0$. Then we have the elementary symmetric functions

$$\begin{aligned}\sum \alpha &= \alpha+\beta+\gamma+\delta = -p \\ \sum \alpha\beta &= \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = q \\ \sum \alpha\beta\gamma &= \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -r \\ \sum \alpha\beta\gamma\delta &= \alpha\beta\gamma\delta = s\end{aligned}$$

Any other symmetric functions can be written in terms of these functions. Some functions are the consequences of well known formulas and some require logical thinking.

1. $\sum \frac{1}{\alpha}$

$$\begin{aligned}\sum \frac{1}{\alpha} &= \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta} \\ &= \frac{\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma}{\alpha\beta\gamma\delta} \\ &= \frac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta}\end{aligned}$$

2. $\sum \alpha^2$

$\sum \alpha^2 = \alpha^2+\beta^2+\gamma^2+\delta^2$ is a part of the expansion $(\alpha+\beta+\gamma+\delta)^2$.

$$\begin{aligned}(\alpha+\beta+\gamma+\delta)^2 &= \alpha^2+\beta^2+\gamma^2+\delta^2+2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) \\ \alpha^2+\beta^2+\gamma^2 &= (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) \\ \sum \alpha^2 &= \left(\sum \alpha\right)^2 - 2\sum \alpha\beta\end{aligned}$$

3. $\sum \alpha^2\beta\gamma$

$\sum \alpha^2\beta\gamma = \alpha^2\beta\gamma + \alpha^2\beta\delta + \alpha^2\gamma\delta + \beta^2\alpha\gamma + \beta^2\alpha\delta + \beta^2\gamma\delta + \gamma^2\alpha\beta + \gamma^2\alpha\delta + \gamma^2\beta\delta + \delta^2\alpha\beta + \delta^2\alpha\gamma + \delta^2\beta\gamma$. This is a part of the product $(\sum \alpha)(\sum \alpha\beta\gamma)$

$$\begin{aligned}\left(\sum \alpha\right)\left(\sum \alpha\beta\gamma\right) &= (\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta) \\ &= \sum \alpha^2\beta\gamma + 4\alpha\beta\gamma\delta \\ \sum \alpha^2\beta\gamma &= \left(\sum \alpha\right)\left(\sum \alpha\beta\gamma\right) - 4\alpha\beta\gamma\delta\end{aligned}$$

4. $\sum \alpha^2\beta^2$

$\sum \alpha^2\beta^2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2$ is a part of the expansion $(\sum \alpha\beta)^2$.

$$\begin{aligned}\left(\sum \alpha\beta\right)^2 &= \left(\sum \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta\right)^2\\ &= \sum \alpha^2\beta^2 + 2\left[\alpha^2\beta\gamma + 3\alpha\beta\gamma\delta\right]\\ \sum \alpha^2\beta^2 &= \left(\sum \alpha\beta\right)^2 - 2\alpha^2\beta\gamma - 6\alpha\beta\gamma\delta\end{aligned}$$

5. $\sum \alpha^4$

$\sum \alpha^4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4$ is a part of the expansion $(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^2$.

$$\begin{aligned}\left(\alpha^2 + \beta^2 + \gamma^2 + \delta^2\right)^2 &= \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + 2\sum \alpha^2\beta^2\\ &= \sum \alpha^4 + 2\sum \alpha^2\beta^2\\ \sum \alpha^4 &= \left(\sum \alpha^2\right)^2 - 2\sum \alpha^2\beta^2\end{aligned}$$

Problem 4.2.1: If α, β and γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, express the value of $\sum \alpha^2\beta$ in terms of the coefficients.

Solution: We have,

$$\sum \alpha = -p$$

$$\sum \alpha\beta = q$$

$$\alpha\beta\gamma = -r$$

$$\begin{aligned}\sum \alpha^2\beta &= \alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta\\ &= \alpha^2\beta + \beta^2\alpha + \beta^2\gamma + \gamma^2\beta + \gamma^2\alpha + \alpha^2\gamma\\ &= \alpha\beta(\alpha+\beta) + \beta\gamma(\beta+\gamma) + \alpha\gamma(\alpha+\gamma)\\ &= \alpha\beta(-p-\gamma) + \beta\gamma(p-\alpha) + \alpha\gamma(-p-\beta) \quad \left[\text{Using } \sum \alpha = \alpha+\beta+\gamma = -p\right]\\ &= -p\alpha\beta - \alpha\beta\gamma - p\beta\gamma - \alpha\beta\gamma - p\alpha\gamma - \alpha\beta\gamma\\ &= -p(\alpha\beta + \beta\gamma + \alpha\gamma) - 3\alpha\beta\gamma\\ &= -pq - 3(-r) \quad \left[\text{Using } \sum \alpha\beta = q\right]\\ &= -pq + 3r\end{aligned}$$

Problem 4.2.2: If α, β and γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, prove that $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = r - pq$.

Solution: We have,

$$\sum \alpha = -p$$

$$\sum \alpha\beta = q$$

$$\alpha\beta\gamma = -r$$

$$\begin{aligned}
(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) &= (\alpha+\beta+\gamma-\gamma)(\alpha+\beta+\gamma-\alpha)(\alpha+\beta+\gamma-\beta)\\
&= (-p-\gamma)(-p-\alpha)(-p-\beta)\\
&= -(p+\alpha)(p+\beta)(p+\gamma)\\
&= -\left[p^3+(\alpha+\beta+\gamma)p^2+(\alpha\beta+\beta\gamma+\gamma\alpha)p+\alpha\beta\gamma\right]\\
&= -\left[p^3+(-p)p^2+qp-r\right]\\
&= r-pq
\end{aligned}$$

Problem 4.2.3: If α, β and γ are the roots of the equation $x^3+px^2+qx+r=0$, find the values of (i) $\sum\frac{1}{\alpha}$, (ii) $\sum\frac{1}{\alpha\beta}$, (iii) $\sum\alpha^2$, (iv) $\sum\frac{1}{\alpha^2}$

Solution: We have,

$$\sum\alpha = -p$$

$$\sum\alpha\beta = q$$

$$\alpha\beta\gamma = -r$$

$$\begin{aligned}
(i)\quad \sum\frac{1}{\alpha} &= \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\\
&= \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}\\
&= \frac{\sum\alpha\beta}{\alpha\beta\gamma}\\
&= -\frac{q}{r}\\
(ii)\quad \sum\frac{1}{\alpha\beta} &= \frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}\\
&= \frac{\gamma+\alpha+\beta}{\alpha\beta\gamma}\\
&= \frac{\sum\alpha}{\alpha\beta\gamma}\\
&= \frac{-p}{-r} = \frac{p}{r}\\
(iii)\quad \sum\alpha^2 &= \alpha^2+\beta^2+\gamma^2\\
&= (\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)\\
&= (-p)^2-2q\\
&= p^2-2q\\
(iv)\quad \sum\frac{1}{\alpha^2} &= \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\\
&= \frac{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}{\alpha^2\beta^2\gamma^2}
\end{aligned}$$

$$= \frac{(\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)}{(\alpha\beta\gamma)^2}$$
$$= \frac{q^2-2r(-p)}{r^2}$$
$$= \frac{q^2-2pr}{r^2}$$

Here are some tricks that you must know...

1. $\sum\alpha^3$ can be got from the product $(\alpha^2+\beta^2+\gamma^2)(\alpha+\beta+\gamma)$. Note that this product contains 9 terms. $(\alpha^2+\beta^2+\gamma^2)(\alpha+\beta+\gamma) = \alpha^3+\beta^3+\gamma^3+\sum\alpha^2\beta$

2. $\sum\alpha^2\beta$ can be got from the product $\sum\alpha\beta\cdot\sum\alpha$. Note that this product contains 9 terms.
$$\begin{aligned}\sum\alpha\beta\cdot\sum\alpha &= (\alpha\beta+\beta\gamma+\alpha\gamma)(\alpha+\beta+\gamma)\\ &= \alpha^2\beta+\alpha\beta^2+\alpha\beta\gamma+\beta\alpha\gamma+\beta^2\gamma+\beta\gamma^2+\alpha^2\gamma+\alpha\beta\gamma+\alpha\gamma^2\\ &= \sum\alpha^2\beta+3\alpha\beta\gamma\\ \sum\alpha^2\beta &= \sum\alpha\beta\cdot\sum\alpha-3\alpha\beta\gamma\end{aligned}$$

Problem 4.2.4: If α, β and γ are the roots of the equation $x^3+px^2+qx+r=0$, find the values of (i) $\sum\alpha^2\beta$, (ii) $\sum\alpha^3$.

Solution: We have,
$$\sum\alpha = -p$$
$$\sum\alpha\beta = q$$
$$\alpha\beta\gamma = -r$$

(i) $\sum\alpha^2\beta$ is a part of the product $(\sum\alpha)(\sum\alpha\beta)$.
$$\begin{aligned}\left(\sum\alpha\right)\left(\sum\alpha\beta\right) &= (\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)\\ &= \sum\alpha^2\beta+3\alpha\beta\gamma\\ \sum\alpha^2\beta &= \left(\sum\alpha\right)\left(\sum\alpha\beta\right)-3\alpha\beta\gamma\\ &= -pq-3r\end{aligned}$$

(ii) $\sum\alpha^3$ is a part of the product $\sum\alpha\cdot\sum\alpha^2$.
$$\begin{aligned}\left(\sum\alpha\right)\left(\sum\alpha^2\right) &= (\alpha+\beta+\gamma)\left(\alpha^2+\beta^2+\gamma^2\right)\\ &= \sum\alpha^3+\sum\alpha^2\beta\\ \sum\alpha^3 &= \left(\sum\alpha\right)\left(\sum\alpha^2\right)-\sum\alpha^2\beta\end{aligned}$$

We know that $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta = p^2-2q$. Hence,
$$\sum\alpha^3 = -p(p^2-2q)-(-pq-3r)$$

$$= -p^3 + 2pq + pq + 3r$$
$$= -p^3 + 3pq + 3r$$

Problem 4.2.5: If α, β γ and δ are the roots of the equation $x^4+px^3+qx^2+rx+s=0$, find (i) $\sum \alpha^2$, (ii) $\sum \alpha^2\beta\gamma$.

Solution: We have,

$$\sum \alpha = -p$$
$$\sum \alpha\beta = q$$
$$\sum \alpha\beta\gamma = -r$$
$$\alpha\beta\gamma\delta = s$$

(i) $\sum \alpha^2$ is a part of the expansion $(\sum \alpha)^2$.

$$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$$
$$= (-p)^2 - 2q$$
$$= p^2 - 2q$$

(ii) $\sum \alpha^2\beta\gamma$ is a part of the product $(\sum \alpha)(\sum \alpha\beta\gamma)$.

$$\left(\sum \alpha\right)\left(\sum \alpha\beta\gamma\right) = (\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$$
$$= \alpha^2\beta\gamma + \alpha^2\beta\delta + \alpha^2\gamma\delta + \beta^2\alpha\gamma + \beta^2\alpha\delta + \beta^2\gamma\delta$$
$$+ \gamma^2\alpha\beta + \gamma^2\alpha\delta + \gamma^2\beta\delta + \delta^2\alpha\beta + \delta^2\alpha\gamma + \delta^2\beta\gamma$$
$$= \sum \alpha^2\beta\gamma + 4\alpha\beta\gamma\delta$$
$$\sum \alpha^2\beta\gamma = \left(\sum \alpha\right)\left(\sum \alpha\beta\gamma\right) - 4\alpha\beta\gamma\delta$$
$$= (-p)(-r) - 4s$$
$$= pr - 4s$$

Problem 4.2.6: If α, β γ and δ are the roots of the equation $x^4+px^3+qx^2+rx+s=0$, find (i) $\sum \alpha^2\beta^2$, (ii) $\sum \alpha^3\beta$ and (iii) $\sum \alpha^4$.

Solution: We have,

$$\sum \alpha = -p$$
$$\sum \alpha\beta = q$$
$$\sum \alpha\beta\gamma = -r$$
$$\alpha\beta\gamma\delta = s$$

(i) $\sum \alpha^2\beta^2$ is a part of the expansion $(\sum \alpha\beta)^2$.

$$\left(\sum \alpha\beta\right)^2 = (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)^2$$
$$= \alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2$$
$$+ 2\left[\alpha^2\beta\gamma + \alpha^2\beta\delta + \beta^2\alpha\gamma + \beta^2\alpha\delta + \alpha\beta\gamma\delta\right.$$

$$+\alpha^2\gamma\delta+\gamma^2\alpha\beta+\alpha\beta\gamma\delta+\gamma^2\alpha\delta+\alpha\beta\gamma\delta$$
$$+\delta^2\alpha\beta+\delta^2\alpha\gamma+b^2\gamma\delta+\gamma^2\beta\delta+\delta^2\beta\gamma\big]$$
$$=\sum\alpha^2\beta^2+2\left[\alpha^2\beta\gamma+3\alpha\beta\gamma\delta\right]$$
$$=\sum\alpha^2\beta^2+2\alpha^2\beta\gamma+6\alpha\beta\gamma\delta$$
$$\sum\alpha^2\beta^2=\left(\sum\alpha\beta\right)^2-2\alpha^2\beta\gamma-6\alpha\beta\gamma\delta \qquad (1)$$

We have, $\sum\alpha^2\beta\gamma=(\sum\alpha)(\sum\alpha\beta\gamma)-4\alpha\beta\gamma\delta=pr-4s$. Substituting in (1),

$$\sum\alpha^2\beta^2=q^2-2\left[pr-4s\right]-6s$$
$$=q^2-2pr+2s$$

(*ii*) $\sum\alpha^3\beta$ is a part of the product $(\sum\alpha^2)(\sum\alpha\beta)$

$$\left(\sum\alpha^2\right)\left(\sum\alpha\beta\right)=\left(\alpha^2+\beta^2+\gamma^2+\delta^2\right)(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$$
$$=\alpha^3\beta+\alpha^3\gamma+\alpha^3\delta+\alpha^2\beta\gamma+\alpha^2\beta\delta+\alpha^2\gamma\delta$$
$$+\beta^3\alpha+\beta^2\alpha\gamma+\beta^2\alpha\delta+\beta^3\gamma+\beta^3\delta+\beta^2\gamma\delta$$
$$+\gamma^2\alpha\beta+\gamma^3\alpha+\gamma^2\alpha\delta+\gamma^3\beta+\gamma^2\beta\delta+\gamma^3\delta$$
$$+\delta^2\alpha\beta+\delta^2\alpha\gamma+\delta^3\alpha+\delta^2\beta\gamma+\delta^3\beta+\delta^3\gamma$$
$$=\sum\alpha^3\beta+\sum\alpha^2\beta\gamma$$
$$\sum\alpha^3\beta=\left(\sum\alpha^2\right)\left(\sum\alpha\beta\right)-\sum\alpha^2\beta\gamma \qquad (2)$$

We have, $\sum\alpha^2\beta\gamma=pr-4s$ and $\sum\alpha^2=p^2-2q$. Substituting in (2),

$$\sum\alpha^3\beta=\left(p^2-2q\right)q-(pr-4s)$$
$$=p^2q-2q^2-pr+4s$$

(*iii*) $\sum\alpha^4=\alpha^4+\beta^4+\gamma^4+\delta^4$ is a part of the expansion $(\alpha^2+\beta^2+\gamma^2+\delta^2)^2$.

$$\left(\alpha^2+\beta^2+\gamma^2+\delta^2\right)^2=\alpha^4+\beta^4+\gamma^4+\delta^4+2\sum\alpha^2\beta^2$$
$$=\sum\alpha^4+2\sum\alpha^2\beta^2$$
$$\sum\alpha^4=\left(\sum\alpha^2\right)^2-2\sum\alpha^2\beta^2 \qquad (3)$$

We have, $\sum\alpha^2=p^2-2q$ and $\sum\alpha^2\beta^2=q^2-2pr+2s$. Substituting in (3),

$$\sum\alpha^4=\left(p^2-2q\right)^2-2(q^2-2pr+2s)$$
$$=p^4-4p^2q+4pr-2q^2-4s$$

Problem 4.2.7: If α, β γ and δ are the roots of the equation $x^4-15x^2+10x+24=0$, find the values of (*i*) $\sum\alpha^2\beta$, (*ii*) $\sum\frac{1}{\alpha^2}$.

Solution: We have,

$$\sum\alpha=0$$
$$\sum\alpha\beta=-15$$
$$\sum\alpha\beta\gamma=-10$$

$\alpha\beta\gamma\delta = 24$

(i) $\sum \alpha^2\beta$ is a part of the product $(\sum \alpha)(\sum \alpha\beta)$.

$$\left(\sum \alpha^2\right)\left(\sum \alpha\beta\right) = (\alpha+\beta+\gamma+\delta)(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$$
$$= \alpha^2\beta+\alpha^2\gamma+\alpha^2\delta+\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta$$
$$+\beta^2\alpha+\beta\alpha\gamma+\beta 2\alpha\delta+\beta^2\gamma+\beta^2\delta+\beta\gamma\delta$$
$$+\gamma\alpha\beta+\gamma^2\alpha+\gamma\alpha\delta+\gamma^2\beta+\gamma\beta\delta+\gamma^2\delta$$
$$+\delta\alpha\beta+\delta\alpha\gamma+\delta^2\alpha+\delta\beta\gamma+\delta^2\beta+\delta^2\gamma$$
$$= \sum \alpha^2\beta + \sum \alpha\beta\gamma$$
$$\sum \alpha^2\beta = \left(\sum \alpha\right)\left(\sum \alpha\beta\right) - \sum \alpha\beta\gamma \qquad (1)$$

Substituting the values in (1),

$$\sum \alpha^2\beta = 0 \times (-15\_9 - 10)$$
$$= 10$$

(ii) $\sum \frac{1}{\alpha^2}$ is a part of the expansion $\left(\sum \frac{1}{\alpha}\right)^2$.

$$\sum \frac{1}{\alpha^2} = \left(\sum \frac{1}{\alpha}\right)^2 - 2\sum \frac{1}{\alpha}\frac{1}{\beta}$$
$$= \left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)^2 - 2\left[\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\alpha\delta}+\frac{1}{\beta\gamma}+\frac{1}{\beta\delta}+\frac{1}{\gamma\delta}\right]$$
$$= \left(\frac{\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma}{\alpha\beta\gamma\delta}\right)^2 - 2\left[\frac{\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta}{\alpha\beta\gamma\delta}\right]$$
$$= \left(\frac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta}\right)^2 - 2\left(\frac{\sum \alpha\beta}{\alpha\beta\gamma\delta}\right)^2$$
$$= \left(\frac{-10}{24}\right)^2 - 2\left(\frac{-15}{24}\right)^2 = \frac{5}{16}$$

Problem 4.2.8: If α, β γ and δ are the roots of the equation $x^4+qx^2+rx+s=0$, find the value of $\sum \frac{\alpha+\beta+\gamma-\delta}{2\delta^2}$.

Solution: We have,

$$\sum \alpha = 0$$
$$\sum \alpha\beta = q$$
$$\sum \alpha\beta\gamma = -r$$
$$\alpha\beta\gamma\delta = s$$
$$\sum \frac{\alpha+\beta+\gamma-\delta}{2\delta^2} = \sum \frac{\alpha+\beta+\gamma+\delta-2\delta}{2\delta^2}$$
$$= \sum \frac{0-2\delta}{2\delta^2}$$

$$= \sum \frac{-2\delta}{2\delta^2}$$
$$= -\sum \frac{1}{\delta}$$
$$= -\left[\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right]$$
$$= -\left[\frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta}\right]$$
$$= -\frac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta}$$
$$= -\frac{-r}{s}$$
$$= \frac{r}{s}$$

Problem 4.2.9: If α, β and γ are the roots of the equation $x^3 + qx + r = 0$, find the values of (i) $\sum \alpha^2\beta^2$, (ii) $\sum \frac{\alpha}{\beta\gamma}$, (iii) $\sum \frac{\alpha\beta}{\gamma}$, (iv) $\sum \alpha^2\beta$, (v) $\sum \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$, (vi) $\sum \frac{1}{\beta+\gamma}$, and (vii) $\sum \frac{\alpha}{\beta+\gamma}$.

Solution: We have,

$$\sum \alpha = 0$$
$$\sum \alpha\beta = q$$
$$\alpha\beta\gamma = -r$$

(i) $\sum \alpha^2\beta^2$ is a part of the expansion of $(\sum \alpha\beta)^2$.

$$\sum \alpha^2\beta^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$$
$$= (a\beta + \beta\gamma + \gamma\alpha)^2 - 2\left(\alpha\beta^2\gamma + \beta\gamma^2\alpha + \alpha^2\beta\gamma\right)$$
$$= (a\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$
$$= \left(\sum \alpha\beta\right)^2 - 2\alpha\beta\gamma\left(\sum \alpha\right)$$
$$= q^2 - 2(-r) \times 0$$
$$= q^2$$

(ii) $\sum \frac{\alpha}{\beta\gamma}$

$$\sum \frac{\alpha}{\beta\gamma} = \frac{\alpha}{\beta\gamma} + \frac{\beta}{\alpha\gamma} + \frac{\gamma}{\alpha\beta}$$
$$= \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha\beta\gamma}$$
$$= \frac{(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)}{\alpha\beta\gamma}$$
$$= \frac{0 - 2q}{-r}$$

$$= \frac{2q}{r}$$

(iii) $\sum \frac{\alpha\beta}{\gamma}$

$$\begin{aligned}
\sum \frac{\alpha\beta}{\gamma} &= \frac{\alpha\beta}{\gamma} + \frac{\alpha\gamma}{\beta} + \frac{\beta\gamma}{\alpha} \\
&= \frac{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2}{\alpha\beta\gamma} \\
&= \frac{q^2}{-r} \qquad \text{[From (i)]} \\
&= -\frac{q^2}{r}
\end{aligned}$$

(iv) $\sum \alpha^2\beta$ is a part of the product $(\sum \alpha)(\sum \alpha\beta)$.

$$\begin{aligned}
\left(\sum \alpha\right)\left(\sum \alpha\beta\right) &= (\alpha+\beta+\gamma)(a\beta+\beta\gamma+\gamma\alpha) \\
&= \sum \alpha^2\beta + 3\alpha\beta\gamma \\
\sum \alpha^2\beta &= \left(\sum \alpha\right)\left(\sum \alpha\beta\right) - 3\alpha\beta\gamma \\
&== -3(-r) = 3r
\end{aligned}$$

(v) $\sum \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$

$$\begin{aligned}
\sum \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) &= \sum \frac{\alpha^2+\beta^2}{\alpha\beta} \\
&= \frac{\alpha^2+\beta^2}{\alpha\beta} + \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\alpha^2+\gamma^2}{\alpha\gamma} \\
&= \frac{\alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + a^2\beta + \gamma^2\beta}{\alpha\beta\gamma} \\
&= \frac{\alpha^2\beta}{\alpha\beta\gamma} \\
&= \frac{3r}{-r} \qquad \text{[From (iv), } \sum \alpha^2\beta = 3r\text{]} \\
&= -3
\end{aligned}$$

(vi) $\sum \frac{1}{\beta+\gamma}$

Since $\alpha+\beta+\gamma=0$, we get $\beta+\gamma=-\alpha$. Now the problem is,

$$\begin{aligned}
\sum \frac{1}{\beta+\gamma} &= \sum \frac{1}{-a} \\
&= -\left[\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right] \\
&= -\frac{a\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} \\
&= -\frac{\sum \alpha\beta}{\alpha\beta\gamma}
\end{aligned}$$

$$= -\frac{q}{-r}$$
$$= \frac{q}{r}$$

(*vii*) $\sum \frac{\alpha}{\beta+\gamma}$

Since $\alpha+\beta+\gamma=0$, we get $\beta+\gamma=-\alpha$. Now the problem is,

$$\sum \frac{\alpha}{\beta+\gamma} = \sum \frac{\alpha}{-a}$$
$$= \sum(-1)$$
$$= -3$$

Chapter 5

Transformation of Equations

Let $f(x) = 0$ be a polynomial equation. Without knowing the roots of the equation $f(x) = 0$, we can transform the given equation into another equation $g(x) = 0$whose roots are related to the roots of the first equation $f(x) = 0$ in some way. In this section, we are going to study some important such transformations.

5.1 Multiplying the roots by k

To form an equation whose roots are k-times the roots of a given equation. Let the given equation be $f(x) = 0$ where

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \quad \cdots\cdots (1)$$

Let $\alpha_1, \alpha_2, \cdots \alpha_n$ be the roots of the equation $f(x) = 0$. then,

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n) \quad \cdots\cdots (2)$$

The transformation is $y = kx$. From the transformation we get $x = \frac{y}{k}$. Put $x = \frac{y}{k}$ in (2), we get

$$f\left(\frac{y}{k}\right) = a_0\left(\frac{y}{k} - \alpha_1\right)\left(\frac{y}{k} - \alpha_2\right)\cdots\left(\frac{y}{k} - \alpha_n\right) \quad \cdots\cdots (3)$$

One can easily verify that $k\alpha_1, k\alpha_2, \cdots k\alpha_n$ are the roots of the equation $f\left(\frac{y}{k}\right) = 0$. Hence the required equation is

$$f\left(\frac{y}{k}\right) = 0$$

$$a_0\left(\frac{y}{k}\right)^n + a_1\left(\frac{y}{k}\right)^{n-1} + a_2\left(\frac{y}{k}\right)^{n-2} + \cdots + a_{n-1}\left(\frac{y}{k}\right) + a_n = 0$$

That is,

$$a_0y^n + ka_1y^{n-1} + k^2a_2y^{n-2} + \cdots + k^{n-1}a_{n-1}y + k^na_n = 0$$

Changing the variable to x, the required equation is $a_0x^n + ka_1x^{n-1} + k^2a_2x^{n-2} + \cdots + k^{n-1}a_{n-1}x + k^na_n = 0$.

Working rule:

To obtain the equation whose roots are k times the roots of a given equation, we have to multiply the coefficients of $x^n, x^{n-1}, x^{n-2}, \cdots x$ and the constant term by 1, k, k^2,$\cdots$, k^{n-1} and k^n respectively.

Special usage: This transformation is useful for

1. making the leading coefficient of an equation to unity.

2. removing the fractional coefficients

Problem 5.1.1: Find the transformed equation when the roots of the equation $3x^3 - 10x^2 + 9x + 2 = 0$ are multiplied by 3.

Solution: The required equation is obtained by multiplying the coefficients of x^3, x^2, x and constant term by $1, 3, 3^2, 3^3$ respectively. The transformed equation is

$$\begin{aligned}(1 \times 3)x^3 - (3 \times 10)x^2 + (3^2 \times 9)x + 3^3 \times 2 &= 0\\ 3x^3 - 30x^2 + 81x + 54 &= 0\\ x^3 - 10x^2 + 27x + 18 &= 0 \quad \text{[Dividing by 3]}\end{aligned}$$

Problem 5.1.2: Remove the fractional coefficient from the equation $x^3 + \frac{1}{4}x^2 - \frac{1}{16}x + \frac{1}{72} = 0$

Solution: Let us multiply the roots of the equation by m. The transformed equation is $1 \times x^3 - m \times \frac{1}{4}x^2 + m^2 \times \frac{1}{16}x + m^3 \times \frac{1}{72} = 0$.

We must choose m such that $\frac{m}{4}, \frac{m^2}{16}$ and $\frac{m^3}{72}$ are integers.

Let us find the cube nearest to and greater than 72.

When $m = 12$, all the fractions will be removed. Hence the equation is,

$$\begin{aligned}1 \times x^3 - m \times \frac{1}{4}x^2 + m^2 \times \frac{1}{16}x + m^3 \times \frac{1}{72} &= 0\\ x^3 - 12 \times \frac{1}{4}x^2 + 12^2 \times \frac{1}{16}x + 12^3 \times \frac{1}{72} &= 0\\ x^3 - \frac{12}{4}x^2 + \frac{144}{16}x + \frac{1728}{72} &= 0\\ x^3 - 3x^2 + 9x + 24 &= 0\end{aligned}$$

Problem 5.1.3: Transform the equation $2x^3 - 3x^2 + 5x - 1 = 0$ so that the leading coefficient is unity.

Solution: We should make the coefficient of x^3 as 1.
We multiply the roots of the given equation by 2. Hence the equation is,

$$\begin{aligned}(1 \times 2)x^3 - (2 \times 3)x^2 + (2^2 \times 5)x - 2^3 \times 1 &= 0\\ 2x^3 - 6x^2 + 20x - 8 &= 0\\ x^3 - 3x^2 + 10x - 4 &= 0 \quad \text{[Dividing by 2]}\end{aligned}$$

5.2 Changing the sign of the roots

This is the special case of 1.5.1 when $k = -1$.

Working rule: To obtain the equation whose roots are negative of the roots of a given equation, we have to multiply the coefficients of $x^n, x^{n-1}, x^{n-2}, \cdots x$ and the constant term alternatively by 1 and -1.

Problem 5.2.1: Form an equation whose roots are the negatives of the roots of the equation $4x^5 - 7x^4 + x^3 - 3x^2 + x - 9 = 0$.

Solution: By multiplying the coefficients successively by 1, -1, 1, -1 we obtain the required equation The required equation is, $4x^5 + 7x^4 + x^3 + 3x^2 + x + 9 = 0$.

Problem 5.2.2: Form an equation whose roots are the negatives of the roots of the equation $x^4 + 5x^2 + x - 9 = 0$.

Solution: Here the x^3 term is missing. In that case we write the equation with coefficient 0 for the missing terms. Hence the given equation is $x^4 + 0x^3 + 5x^2 + x - 9 = 0$. By multiplying the coefficients successively by 1, -1, 1, -1 we obtain the required equation The required equation is,

$$x^4 - 0x^3 + 5x^2 - x - 9 = 0$$
$$x^4 + 5x^2 - x - 9 = 0$$

5.3 Reciprocals of the roots

To form an equation whose roots are reciprocals of the roots of a given equation.

Let the given equation be $f(x) = 0$ where

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \quad \cdots\cdots (1)$$

Let $\alpha_1, \alpha_2, \cdots \alpha_n$ be the roots of the equation $f(x) = 0$. then,

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n \quad \cdots\cdots (2)$$

The transformation is $y = \frac{1}{x}$.

From the transformation we get $x = \frac{1}{y}$. Put $x = \frac{1}{y}$ in (2), we get

$$f\left(\frac{1}{y}\right) = a_0\left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right)\cdots\left(\frac{1}{y} - \alpha_n\right) \quad \cdots\cdots (3)$$

One can easily verify that $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \cdots \frac{1}{\alpha_n}$ are the roots of the equation $f\left(\frac{1}{y}\right) = 0$. Hence the required equation is

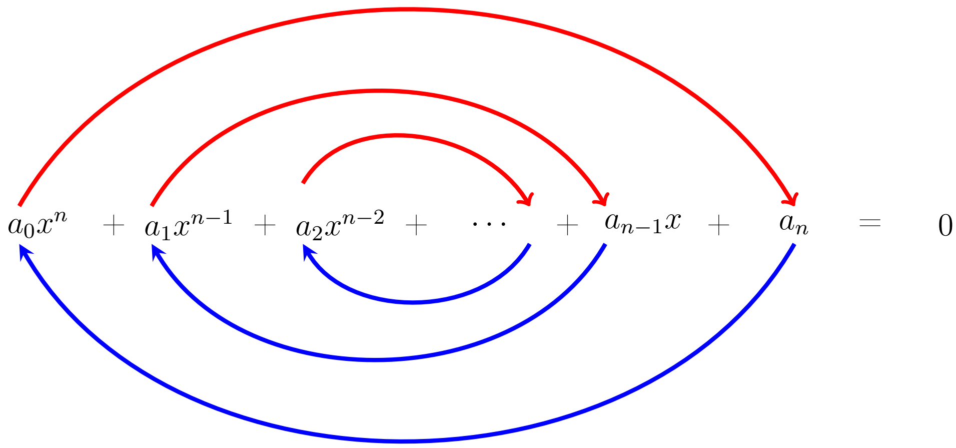
$$f\left(\frac{1}{y}\right) = 0$$
$$a_0\left(\frac{1}{y}\right)^n + a_1\left(\frac{1}{y}\right)^{n-1} + a_2\left(\frac{1}{y}\right)^{n-2} + \cdots + a_{n-1}\left(\frac{1}{y}\right) + a_n = 0$$

That is,

$$\frac{a_0}{y^n} + \frac{a_1}{y^{n-1}} + \frac{a_2}{y^{n-2}} + \cdots + \frac{a_{n-1}}{y} + a_n = 0$$
$$a_0 + a_1y + a_2y^2 + \cdots + a_ny^n = 0$$

Hence the required equation is $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0 = 0$.

Working Rule: We obtain the required equation, by replacing the coefficients in the reverse order.



Problem 5.3.1: Form an equation whose roots are the reciprocals of the roots of the equation $3x^4 + 2x^3 - x^2 + 7x - 1 = 0$.

Solution: First check whether any term is missing in the equation. Here, the given equation is $3x^4 + 2x^3 - x^2 + 7x - 1 = 0$. We obtain the required equation, by replacing the coefficients in the reverse order. The required equation is,

$$-x^4 + 7x^3 - x^2 + 2x + 3 = 0$$
$$x^4 - 7x^3 + x^2 - 2x - 3 = 0 \quad \text{[Dividing by -1]}$$

Problem 5.3.2: Form an equation whose roots are the reciprocals of the roots of the equation $x^4 - 5x^3 + x^2 - 7 = 0$.

Solution: First check whether any term is missing in the equation. Here, the x-term is missing. The given equation is $x^4 - 5x^3 + x^2 + 0x + 7 = 0$.
We obtain the required equation, by replacing the coefficients in the reverse order.
The required equation is,

$$7x^4 + 0x^3 + x^2 - 5x + 1 = 0$$
$$7x^4 + x^2 - 5x + 1 = 0$$

5.4 Diminishing the roots by a given quantity

To form an equation whose roots are decreased by a constant quantity h.

Let the given equation be $f(x) = 0$ where
$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \quad \cdots\cdots (1)$$
Let $\alpha_1, \alpha_2, \cdots \alpha_n$ be the roots of the equation $f(x) = 0$. then,
$$f(x) = a_0(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n \quad \cdots\cdots (2)$$
Decrease the roots by h, Therefore, the transformation is $y = x - h$.
From the transformation we get $x = y + h$.

Put $x = y + h$ in (2), we get

$$f(y+h) = a_0\,(y+h-\alpha_1)\,(y+h-\alpha_2)\cdots(y+h-\alpha_n)$$
$$= a_0\,(y-(\alpha_1-h))\,(y-(\alpha_2-h))\cdots(y-(\alpha_n-h)) \quad \cdots\cdots(3)$$

One can easily verify that the roots of $f(y+h)=0$ are α_1-h, α_2-h, α_3-h, $\cdots$, α_n-h.

From (1), the required equation is $a_0(y+h)^n+a_1(y+h)^{n-1}+a_2(y+h)^{n-2}+\cdots+a_n=0$.

On simplifying using binomial expansion, this equation can be written as

$$b_0y^n + b_1y^{n-1} + b_2y^{n-2} + \cdots + b_n = 0 \quad \cdots\cdots(4)$$

Note that the roots of (4) are $\alpha_1 - h$, $\alpha_2 - h$, $\alpha_3 - h$, $\cdots$, $\alpha_n - h$.

Now we have to find the coefficients b_0, b_1, b_2, $\cdots$, b_n to determine the required equation.

How to find the coefficients of the required equation?

Let us increase the roots of (4) by h to obtain the original equation. We use the transformation $x = y + h$. So, we have $y = x - h$ and the equation becomes

$$b_0(x-h)^n + b_1(x-h)^{n-1} + b_2(x-h)^{n-2} + \cdots + b_n = 0 \quad \cdots\cdots(5)$$

The roots of this equation are $\alpha_1, \alpha_2. \cdots \alpha_n$. Therefore, equations (1) and (5) represent the same equation.

Dividing equation (5) continuously by $(x\breve{\ }h)$, we obtain the remainders as b_0, b_1, b_2, $\cdots$, b_n. Substituting these in (4), we obtain the required equation.

Working rule: To obtain the equation whose roots are diminished by h, divide the equation continuously by $x - h$ and note the remainders b_0, b_1, b_2, $\cdots$, b_n. Then the required equation is $b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0$.

Problem 5.4.1: Diminish the roots of the equation $x^4 - 5x^3 + 7x^2 - 4x + 5 = 0$ by 4.

Solution: To find the required equation, we divide the given equation successively by $x - 4$. We use synthetic division method to find the remainder in each step.

4	1	-5	7	-4	5
	0	4	14	12	32
	1	-1	3	8	37
	0	4	12	60	
	1	3	15	68	
	0	8	28		
	1	7	43		
	0	4			
	1	11			

Hence the required equation is $x^4 + 11x^3 + 43x^2 + 68x + 37 = 0$.

5.5 Increasing the roots by a given quantity

Increasing the roots of an equation by h is equivalent to decreasing the roots by $-h$. Hence instead of dividing the equation continuously by $x - h$, we divide by $x + h$.

Working rule: To obtain the equation whose roots are increased by h, divide the equation continuously by $x + h$ and note the remainders b_0, b_1, b_2, $\cdots$, b_n. Then the required equation is $b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0$.

Problem 5.5.1: Increase the roots of the equation $3x^4 + 7x^3 - 15x^2 + x - 2 = 0$ by 7.

Solution: Increasing the roots by 7 is equivalent to decreasing the roots by ˘7.

To find the required equation, we divide the given equation successively by $x + 7$. We use synthetic division method to find the remainder in each step.

-7	3	7	-15	1	-2
	0	-21	98	-581	4060
	3	-14	83	-580	4058
	0	-21	245	-2296	
	3	-35	328	-2876	
	0	-21	392		
	3	-56	720		
	0	-21			
	3	-77			

Hence the required equation is $3x^4 - 77x^3 + 720x^2 - 2876x + 4058 = 0$.

Problem 5.5.2: Increase the roots of the equation $x^4 - x^3 - 10x^2 + 4x + 24 = 0$ by 2.

Solution: Increasing the roots by 2 is equivalent to decreasing the roots by ˘2.
To find the required equation, we divide the given equation successively by $x + 2$. We use synthetic division method to find the remainder in each step.

-2	1	-1	-10	4	24
	0	-2	6	8	-24
	1	-3	-4	12	0
	0	-2	10	-12	
	1	-5	6	0	
	0	-2	14		
	1	-7	20		
	0	-2			
	1	-9			

Hence the required equation is $x^4 - 9x^3 + 20x^2 = 0$.

5.6 Removing a specific term

To form an equation in which certain specified terms of the given equation are absent. 2mm] Let the given equation be $f(x) = 0$ where

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \quad \cdots\cdots(1)$$

We can remove any term by diminishing the roots of the equation by a certain quantity.

Let us diminish the roots by h. Then the equation becomes

$$a_0(y+h)^n + a_1(y+h)^{n-1} + a_2(y+h)^{n-2} + \cdots + a_n = 0$$

$$a_0\left(y^n + ny^{n-1}h + \frac{n(n-1)}{2}y^{n-2}h^2 + \cdots + h^n\right)$$

$$+a_1\left(y^{n-1} + (n-1)y^{n-2}h + \frac{(n-1)(n-2)}{2}y^{n-3}h^2 + \cdots + h^{n-1}\right)$$

$$+a_2\left(y^{n-2} + (n-2)y^{n-3}h + \frac{(n-2)(n-3)}{2}y^{n-4}h^2 + \cdots + h^{n-2}\right) + \cdots + a_n = 0$$

$$a_0y^n + (a_0nh + a_1)y^{n-1} + \left(a_0\frac{n(n-1)}{2}h^2 + a_1(n-1)h + a_2\right)y^{n-2} + \cdots + a_n = 0$$

- If we want to remove the second term, we must have

  $$a_0nh + a_1 = 0$$
  $$h = -\frac{a_1}{na_0}$$

- If we want to remove the second term, we must have

  $$a_0\frac{n(n-1)}{2}h^2 + a_1(n-1)h + a_2 = 0$$

  By solving this equation for h, we get two values for h and reducing the roots by$h$ we get the equations.

Problem 5.6.1: Remove the second term of $x^4 - 12x^3 + 48x^2 - 70x + 35 = 0$.

Solution: To remove the second term, we must diminish the roots by h where

$$h = -\frac{-12}{4}$$
$$= 3$$

We have to decrease the roots by 3.

3	1	-12	48	-70	35
	0	3	-27	63	-21
	1	-9	21	-7	14
	0	3	-18	9	
	1	-6	3	2	
	0	3	-9		
	1	-3	-6		
	0	3			
	1	0			

The transformed equation is $x^4 - 6x^2 + 2x + 14 = 0$.

Problem 5.6.2: Solve the equation $x^4 + 4x^3 + 5x^2 + 2x - 6 = 0$ by removing the second term.

Solution: To remove the second term, we must diminish the roots by h where

$$h = -\frac{a_1}{na_0}$$
$$= -\frac{4}{4 \times 1}$$
$$= -1$$

We have to increase the roots by 1.

-1	1	4	5	2	-6
	0	-1	-3	-2	0
	1	3	2	0	-6
	0	-1	-2	0	
	1	2	0	0	
	0	-1	-1		
	1	1	-1		
	0	-1			
	1	0			

The transformed equation is $x^4 - x^2 - 6 = 0$.

Now, let us solve the transformed equation.
Put $z = x^2$. The equation becomes,

$$z^2 - z - 6 = 0$$
$$(z-3)(z+2) = 0$$
$$z = 3 \text{ and } -2$$

(i) When $z = 3$:
$\implies x^2 = 3$
$\implies x = \pm\sqrt{3}$

(ii) When $z = -2$:
$\implies x^2 = -2$
$\implies x = \pm i\sqrt{2}$

The roots of the transformed equation are $-\sqrt{3}, \sqrt{3}, -i\sqrt{2}, i\sqrt{2}$.

Note that this equation is obtained by increasing the roots by 1 [diminishing by -1].

Therefore, the roots of the original equation are $-\sqrt{3}-1, \sqrt{3}-1, -i\sqrt{2}-1, i\sqrt{2}-1$.

That is, $-1-\sqrt{3}, -1+\sqrt{3}, -1-i\sqrt{2}, -1+i\sqrt{2}$.

Problem 5.6.3: Transform the equation $x^4 - 4x^3 - 18x^2 - 3x + 2 = 0$ into an equation with the third term absent.

Solution: To remove the third term, we must diminish the roots by h where

$$
\begin{aligned}
a_0\frac{n(n-1)}{2}h^2 + a_1(n-1)h + a_2 &= 0\\
1\times\frac{4\times 3}{2}h^2 - 4\times 3h - 18 &= 0\\
6h^2 - 12h - 18 &= 0\\
h^2 - 2h - 3 &= 0\\
(h-3)(h+1) &= 0\\
h &= 3 \text{ and } -1
\end{aligned}
$$

Case (i): When $h = 3$:

3	1	-4	-18	-3	2
	0	3	-3	63	180
	1	-1	-21	60	182
	0	3	6	-45	
	1	2	-15	15	
	0	3	15		
	1	5	0		
	0	3			
	1	8			

The transformed equation is $x^4 + 8x^3 + 15x + 182 = 0$.

Case (ii): When $h = -1$:

-1	1	-4	-18	-3	2
	0	-1	5	13	-10
	1	-5	-13	10	-8
	0	-1	6	7	
	1	-6	-7	17	
	0	-1	7		
	1	-7	0		
	0	-1			
	1	-8			

The transformed equation is $x^4 - 8x^3 + 17x - 8 = 0$.

5.7 General Transformation

Let $\alpha_1, \alpha_2, \cdots \alpha_n$ are the roots of the polynomial equation $f(x) = 0$. Then finding an equation whose roots are $\phi(\alpha_1), \phi(\alpha_2), \cdots \phi(\alpha_n)$ is known as a general transformation of the given equation. In this case, the relation between a root x of $f(x) = 0$ and a root y of the transformed equation is that $y = \phi(x)$. Also, to obtain this new equation we have to eliminate x between $f(x) = 0$ and $y = \phi(x)$.

Problem 5.7.1: If α, β, γ are the roots of the equation $x^3 - x - 1 = 0$, find the equation with roots $\dfrac{1+\alpha}{1-\alpha}$, $\dfrac{1+\beta}{1-\beta}$ and $\dfrac{1+\gamma}{1-\gamma}$. Hence, show that $\dfrac{1+\alpha}{1-\alpha} + \dfrac{1+\beta}{1-\beta} + \dfrac{1+\gamma}{1-\gamma} = -7$.

Solution: The transformation is, $y = \dfrac{1+\alpha}{1-\alpha} = \dfrac{1+x}{1-x}$.
The given equation is $x^3 - x - 1 = 0 \quad \cdots\cdots \quad (1)$
Put $y = \dfrac{1+x}{1-x}$ in (1)

$$\begin{aligned} y &= \frac{1+x}{1-x} \\ (1-x)y &= 1+x \\ y - xy &= 1+x \\ x + xy &= y-1 \\ x &= \frac{y-1}{y+1} \end{aligned}$$

Substituting in equation (1), we get

$$\left(\frac{y-1}{y+1}\right)^3 - \left(\frac{y-1}{y+1}\right) - 1 = 0$$

$$(y-1)^3 - (y-1)(y+1)^2 - (y+1)^3 = 0$$

$$y^3 - 3y^2 + 3y - 1 - (y-1)\left(y^2 + 2y + 1\right) - \left(y^3 + 3y^2 + 3y + 1\right) = 0$$

$$y^3 - 3y^2 + 3y - 1 - \left(y^3 + 2y^2 + y - y^2 - 2y - 1\right) - \left(y^3 + 3y^2 + 3y + 1\right) = 0$$

$$-y^3 - 7y^2 + y - 1 = 0$$

$$y^3 + 7y^2 - y + 1 = 0$$

This is the transformed equation.
The sum of the roots, $\dfrac{1+\alpha}{1-\alpha} + \dfrac{1+\beta}{1-\beta} + \dfrac{1+\gamma}{1-\gamma} = -7.$

Problem 5.7.2: If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, form the equation whose roots are $\alpha - \dfrac{1}{\beta\gamma}$, $\beta - \dfrac{1}{\alpha\gamma}$ and $\gamma - \dfrac{1}{\alpha\beta}$.

Solution: The given equation is $x^3 - x - 1 = 0 \quad \cdots\cdots \quad (1)$
Next we have to find the general transformation.

$$\begin{aligned} \alpha - \frac{1}{\beta\gamma} &= \alpha - \frac{\alpha}{\alpha\beta\gamma} \\ &= \alpha - \frac{\alpha}{-r} \quad [\text{Since, } S_3 = \alpha\beta\gamma = -r] \\ &= \alpha + \frac{\alpha}{r} \end{aligned}$$

Hence we have to find the equation whose roots are $\alpha + \dfrac{\alpha}{r}$, $\beta + \dfrac{\beta}{r}$ and $\gamma + \dfrac{\gamma}{r}$.

Hence, the transformation is, $y = \alpha + \dfrac{\alpha}{r} = x + \dfrac{x}{r}$.

$$y = x + \frac{x}{r}$$

$$ry = xr + x$$

$$ry = x(1+r)$$

$$x = \frac{yr}{1+r}$$

Substituting in equation (1), we get

$$\left(\frac{yr}{1+r}\right)^3 + p\left(\frac{yr}{1+r}\right)^2 + q\left(\frac{yr}{1+r}\right) + r = 0$$

$$\frac{y^3r^3}{(1+r)^3} + p\frac{y^2r^2}{(1+r)^2} + q\frac{yr}{1+r} + r = 0$$

$$\frac{y^3r^3 + py^2r^2(1+r) + qyr(1+r)^2 + r(1+r)^3}{(1+r)^3} = 0$$

$$r^3y^3 + py^2r^2(1+r) + qyr(1+r)^2 + r(1+r)^3 = 0$$

$$r^3y^3 + pr^2(1+r)y^2 + qr(1+r)^2y + r(1+r)^3 = 0$$

Dividing by r

$$r^2y^3 + pr(1+r)y^2 + q(1+r)^2y + (1+r)^3 = 0$$

This is the required equation whose roots are $\alpha - \frac{1}{\beta\gamma}$, $\beta - \frac{1}{\alpha\gamma}$ and $\gamma - \frac{1}{\alpha\beta}$.

Problem 5.7.3: Find the equation whose roots are the squares of the roots of the equation $x^4 - px^3 + qx^2 - rx + s = 0$.

Solution: The given equation is $x^4 - px^3 + qx^2 - rx + s = 0 \quad \cdots\cdots \quad (1)$ The general transformation is, $y = x^2 \quad \cdots\cdots \quad (2)$

Rearrange the terms of Equation (1) and replace x^2 by y.

$$x^4 + qx^2 + s - \left(px^3 + rx\right) = 0$$

$$\left(x^2\right)^2 + qx^2 + s - x\left(px^2 + r\right) = 0$$

$$y^2 + qy + s - x(py + r) = 0 \quad [\text{Since } x^2 = y]$$

$$y^2 + qy + s = x(py + r)$$

Squaring both sides

$$\left(y^2 + qy + s\right)^2 = x^2\left(py + r\right)^2$$

$$y^4 + q^2y^2 + s^2 + 2qy^3 + 2sy^2 + 2qsy = y\left(p^2y^2 + 2pry + r^2\right)$$

$$y^4 + 2qy^3 + q^2y^2 + 2sy^2 + 2qsy + s^2 = p^2y^3 + 2pry^2 + r^2y$$

$$y^4 + 2qy^3 - p^2y^3 + q^2y^2 + 2sy^2 - 2pry^2 + 2qsy - r^2y + s^2 = 0$$

$$y^4 + (2q - p^2)y^3 + (q^2 + 2s - 2pr)y^2 + (2qs - r^2)y + s^2 = 0$$

The required equation is $x^4 + (2q - p^2)x^3 + (q^2 + 2s - 2pr)x^2 + (2qs - r^2)x + s^2 = 0$.

Problem 5.7.4: Form an equation whose roots are the cubes of the roots of the equation $x^4 - x^3 - 7x^2 + x + 6 = 0$.

Solution: The given equation is $x^4 - x^3 - 7x^2 + x + 6 = 0 \quad \cdots\cdots \quad (1)$

The general transformation is, $y = x^3 \quad \cdots\cdots \quad (2)$

Rearrange the terms of Equation (1) and replace x^3 by y.

$$x^4 + x - x^3 - 7x^2 + 6 = 0$$

$$x\left(x^3 + 1\right) - x^3 - 7x^2 + 6 = 0$$

$$x(y+1) - y - 7x^2 + 6 = 0 [\text{Since } x^3 = y]$$

$$x(y+1) - 7x^2 = y - 6$$

Cubing both sides using the formula $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$,

$$\left[x(y+1) - 7x^2\right]^3 = (y-6)^3$$

$$x^3(y+1)^3 - 7^3x^6 - 3 \cdot x(y+1) \cdot 7x^2\left[x(y+1) - 7x^2\right] = y^3 - 6^3 - 18y(y-6)$$

$$y\left(y^3 + 1 + 3y^2 + 3y\right) - 343y^2 - 21x^3(y+1)(y-6) = y^3 - 216 - 18y^2 + 108y$$

Since, $x(y+1) - 7x^2 = y - 6$ and $x^3 = y$

$$y^4 + y + 3y^3 + 3y^2 - 343y^2 - 21y\left(y^2 - 5y - 6\right) = y^3 - 216 - 18y^2 + 108y$$

$$y^4 + y + 3y^3 + 3y^2 - 343y^2 - 21y^3 + 105y^2 + 126y = y^3 - 216 - 18y^2 + 108y$$

$$y^4 - 19y^3 - 217y^2 + 19y + 216 = 0$$

The required equation is $x^4 - 19x^3 - 217x^2 + 19x + 216 = 0$.

Problem 5.7.5: If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$, form the equation whose roots are $\alpha\beta, \alpha\gamma, \beta\gamma$.

Solution: The given equation is $x^3 + ax^2 + bx + c = 0 \quad \cdots\cdots \quad (1)$

Next we find the general transformation.

$$\begin{aligned} \alpha\beta &= \frac{\alpha\beta\gamma}{\gamma} \\ &= \frac{-c}{\gamma} \quad [\text{Since, } S_3 = \alpha\beta\gamma = -c] \\ &= -\frac{c}{\gamma} \end{aligned}$$

Hence we have to find the equation whose roots are $-\frac{c}{\beta}, -\frac{c}{\alpha}$ and $\frac{c}{\gamma}$.

Hence, the transformation is, $y = -\frac{c}{\gamma} = -\frac{c}{x}$.

$$\begin{aligned} y &= -\frac{c}{x} \\ xy &= -c \\ x &= -\frac{c}{y} \end{aligned}$$

Substituting in equation (1), we get

$$\begin{aligned} \left(-\frac{c}{y}\right)^3 + a\left(-\frac{c}{y}\right)^2 + b\left(-\frac{c}{y}\right) + c &= 0 \\ -\frac{c^3}{y^3} + \frac{ac^2}{y^2} - \frac{bc}{y} + c &= 0 \\ \frac{-c^3 + ac^2y - bcy^2 + cy^3}{y^3} &= 0 \\ cy^3 - bcy^2 + ac^2y - c^3 &= 0 \end{aligned}$$

Dividing by c, we get

$$y^3 - by^2 + acy - c^2 = 0$$

The required equation is $x^3 - bx^2 + acx - c^2 = 0$.

Another method:

We have,

$$\begin{aligned} \alpha + \beta + \gamma &= -a \\ \alpha\beta + \alpha\gamma + \beta\gamma &= b \\ \alpha\beta\gamma &= -c \end{aligned}$$

The new roots are $\alpha\beta, \alpha\gamma, \beta\gamma$.

The required equation is $x^3 - S_1'x^2 + S_2'x - S_3' = 0$, where

$$S_1' = \alpha\beta + \alpha\gamma + \beta\gamma$$

$$\begin{aligned}
&= b \\
S_2' &= \alpha\beta \cdot \alpha\gamma + \alpha\beta \cdot \beta\gamma + \alpha\gamma \cdot \beta\gamma \\
&= \alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 \\
&= \alpha\beta\gamma(\alpha+\beta+\gamma) \\
&= -c \cdot -a \\
&= ac \\
S_3' &= \alpha\beta \cdot \alpha\gamma \cdot \beta\gamma \\
&= (\alpha\beta\gamma)^2 \\
&= c^2
\end{aligned}$$

Hence, the required equation is $x^3 - bx^2 + acx - c^2 = 0$.

Problem 5.7.6: If α is a root of the equation $x^2(x+1)^2 - k(x-1)\left(2x^2+x+1\right) = 0$, prove that $\dfrac{\alpha+1}{\alpha-1}$ is also a root.

Solution: The given equation is $x^2(x+1)^2 - k(x-1)\left(2x^2+x+1\right) = 0 \quad \cdots\cdots \quad (1)$

Let $\alpha/; \beta\ \gamma$ be the roots of equation (1). We find the equation whose roots are $\dfrac{\alpha+1}{\alpha-1}$, $\dfrac{\beta+1}{\beta-1}$ and $\dfrac{\gamma+1}{\gamma-1}$ and show that it is same as equation (1).

The general transformation is $y = \dfrac{x+1}{x-1}$

$$\begin{aligned}
y &= \frac{x+1}{x-1} \\
(x-1)y &= x+1 \\
xy - y &= x+1 \\
xy - x &= y+1 \\
x &= \frac{y+1}{y-1}
\end{aligned}$$

Substituting in equation (1), we get

$$\begin{aligned}
\left(\frac{y+1}{y-1}\right)^2 \left(\frac{y+1}{y-1}+1\right)^2 - k\left(\frac{y+1}{y-1}-1\right)\left[2\left(\frac{y+1}{y-1}\right)^2 + \frac{y+1}{y-1}+1\right] &= 0 \\
\frac{(y+1)^2}{(y-1)^2} \cdot \frac{(2y)^2}{(y-1)^2} - k\frac{2}{y-1} \cdot \frac{2(y+1)^2 + y^2 - 1 + (y-1)^2}{(y-1)^2} &= 0 \\
\frac{4y^2(y+1)^2}{(y-1)^4} - \frac{2k\left[2y^2+4y+2+y^2-1+y^2-2y+1\right]}{(y-1)^2} &= 0 \\
\frac{4y^2(y+1)^2}{(y-1)^4} - \frac{2k\left[4y^2+2y+2\right]}{(y-1)^2} &= 0 \\
\frac{4y^2(y+1)^2 - 2k(y-1)^2\left(4y^2+2y+2\right)}{(y-1)^4} &= 0 \\
4y^2(y+1)^2 - 4k(y-1)^2\left(2y^2+y+1\right) &= 0
\end{aligned}$$

Dividing by 4,

$$y^2(y+1)^2 - k(y-1)^2\left(4y^2+2y+2\right) = 0$$

Hence, the transformed equation is $x^2(x+1)^2 - k(x-1)\left(2x^2+x+1\right) = 0$ which is same as (1).

Therefore, $\dfrac{\alpha+1}{\alpha-1}$ is also a root of the equation (1).

Problem 5.7.7: If α, β, γ are the roots of the equation $2x^3-15x^2+22x+15=0$, form the equation whose roots are (i) $(\alpha-2)^2,\ (\beta-2)^2\ (\gamma-2)^2$, (ii) $\dfrac{1}{(\alpha-2)^2}, \dfrac{1}{(\beta-2)^2}\ \dfrac{1}{(\gamma-2)^2}$.

Solution:

(***i***) The given equation is

$$2x^3-15x^2+22x+15=0 \tag{1}$$

The transformation here is

$$\begin{aligned} y &= (x-2)^2 \\ &= x^2-4x+4 \end{aligned}$$

$$x^2-4x+4-y=0 \tag{2}$$

We have to eliminate x between equations (1) and (2).

(1) $-\,2x\times$ (2) implies

$$-7x^2+14x+2xy-15=0$$

$$-7x^2+(14+2y)x+15=0 \tag{3}$$

Solving equations (eq:2) and (eq:3).

	x^2	x	1	
-4	$4-y$	1	-4	
$2y+14$	15	-7	$2y+14$	

$$\frac{x^2}{-60-(2y+14)(4-y)} = \frac{x}{-7(4-y)-15} = \frac{1}{2y+14-28}$$

$$\frac{x^2}{2y^2+6y-116} = \frac{x}{7y-43} = \frac{1}{2y-14}$$

$$x^2 = \frac{2y^2+6y-116}{2y-14} \tag{4}$$

$$x = \frac{7y-43}{2y-14} \tag{5}$$

From equations (4) and (5),

$$\frac{2y^2+6y-116}{2y-14} = \left(\frac{7y-43}{2y-14}\right)^2$$

$$\frac{2y^2+6y-116}{2y-14} = \frac{(7y-43)^2}{(2y-14)^2}$$

$$\begin{aligned}\left(2y^2+6y-116\right)(2y-14)&=(7y-43)^2\\ 4y^3-28y^2+12y^2-84y-232y+1624&=49y^2-602y+1849\\ 4y^3-65y^2+286y-225&=0\\ 4x^3-65x^2+286x-225&=0 \qquad (6)\end{aligned}$$

Hence, the required equation is $4x^3-65x^2+286x-225=0$.

(***ii***) To form the equation whose roots are $\dfrac{1}{(\alpha-2)^2}, \dfrac{1}{(\beta-2)^2}\ \dfrac{1}{(\gamma-2)^2}$.

The roots of the required equation are the reciprocals of the roots of the equation (6). We obtain the required equation, by replacing the coefficients in the reverse order. The required equation is,

$$\begin{aligned}-225x^3+286x^2-65x+4=0&=0\\ 225x^3-286x^2+65x-4=0&=0 \qquad (7)\end{aligned}$$

(7) is the required equation.

Problem 5.7.8: If α,β,γ are the roots of the equation $x^3-px^2+qx-r=0$, form the equation whose roots are $(i)\ \beta\gamma+\dfrac{1}{\alpha},\ \gamma\alpha+\dfrac{1}{\beta}\ \alpha\beta+\dfrac{1}{\gamma}$, $(ii)\ \dfrac{\alpha}{\beta+\gamma-\alpha}, \dfrac{\beta}{\alpha+\gamma-\beta}, \dfrac{\gamma}{\alpha+\beta-\gamma}$.

Solution:

(***i***) To form the equation whose roots are $\gamma\alpha+\dfrac{1}{\beta}\ \alpha\beta+\dfrac{1}{\gamma}$.

The given equation is

$$x^3-px^2+qx-r=0 \qquad (1)$$

The transformation here is

$$\begin{aligned}y&=\beta\gamma+\frac{1}{\alpha}\\ &=\frac{\alpha\beta\gamma}{\alpha}+\frac{1}{\alpha}\\ &=\frac{\alpha\beta\gamma+1}{\alpha}\end{aligned}$$

Since α,β,γ are the roots of the equation (1), $\alpha\beta\gamma=r$

$$\begin{aligned}y&=\frac{r+1}{\alpha}\\ &=\frac{r+1}{x} \qquad (2)\end{aligned}$$

(2) is the transformation. We solve for x.

$$\begin{aligned}xy&=r+1\\ x&=\frac{r+1}{y} \qquad (3)\end{aligned}$$

Substituting in equation (1),

$$\begin{aligned}\left(\frac{r+1}{y}\right)^3-p\left(\frac{r+1}{y}\right)^2+q\left(\frac{r+1}{y}\right)-r&=0\\ \frac{(r+1)3-p(r+1)^2y+q(r+1)y^2-ry^3}{y^3}&=0\end{aligned}$$

$$ry^3 - q(r+1)y^2 + p(r+1)^2y - (r+1)^3 = 0$$

The required equation is,

$$rx^3 - q(r+1)x^2 + p(r+1)^2x - (r+1)^3 = 0 \tag{4}$$

(***ii***) To form the equation whose roots are $\dfrac{\alpha}{\beta+\gamma-\alpha}, \dfrac{\beta}{\alpha+\gamma-\beta}, \dfrac{\gamma}{\alpha+\beta-\gamma}$.

The transformation here is

$$\begin{aligned} y &= \frac{\alpha}{\beta+\gamma-\alpha} \\ &= \frac{\alpha}{\alpha+\beta+\gamma-2\alpha} \end{aligned}$$

Since α, β, γ are the roots of the equation (1), $\alpha+\beta+\gamma = p$

$$\begin{aligned} y &= \frac{\alpha}{p-2\alpha} \\ &= \frac{x}{p-2x} \end{aligned} \tag{5}$$

(5) is the transformation. We solve for x.

$$\begin{aligned} (p-2x)y &= x \\ py - 2xy &= x \\ x(1+2y) &= py \\ x &= \frac{py}{1+2y} \end{aligned} \tag{6}$$

Substituting in equation (1),

$$\begin{aligned} \left(\frac{py}{1+2y}\right)^3 - p\left(\frac{py}{1+2y}\right)^2 + q\left(\frac{py}{1+2y}\right) - r &= 0 \\ \frac{p^3y^3 - p^3y^2(1+2y) + pqy(1+2y)^2 - r(1+2y)^3}{(1+2y)^3} &= 0 \\ p^3y^3 - p^3y^2(1+2y) + pqy\left(1+4y+4y^2\right) - r\left(1+8y^3+6y+12y^2\right) &= 0 \\ p^3y^3 - p^3y^2 - 2p^3y^3 - pqy + 4pqy^2 + 4pqy^3 - r - 8ry^3 - 6ry - 12ry^2 &= 0 \\ p^3y^3 - p^3y^2(1+2y) + pqy(1+2y)^2 - r(1+2y)^3 &= 0 \\ \left(-p^3 + 4pq - 8r\right)y^3 + \left(-p^3 + 4pq - 12r\right)y^2 + (pq - 6r)y - r &= 0 \\ \left(4pq - 8r - p^3\right)y^3 + \left(4pq - 12r - p^3\right)y^2 + (pq - 6r)y - r &= 0 \end{aligned}$$

The required equation is,

$$\left(4pq - 8r - p^3\right)x^3 + \left(4pq - 12r - p^3\right)x^2 + (pq - 6r)x - r = 0 \tag{7}$$